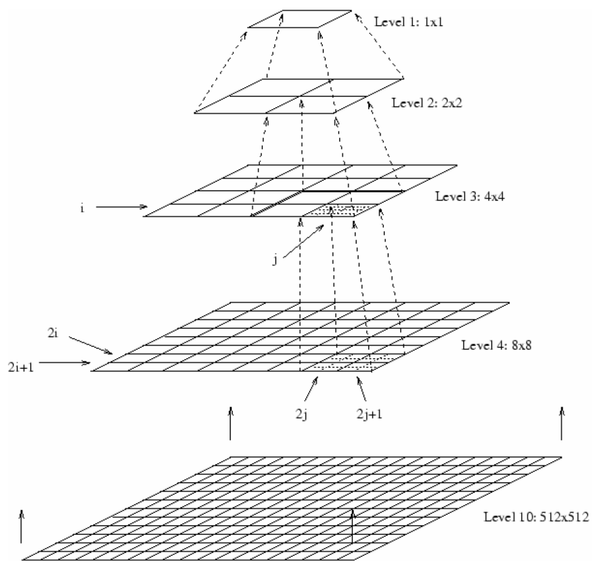
Pyrramids

Lecture-5

Pyramids

- Very useful for representing images.
- Pyramid is built by using multiple copies of image.
- Each level in the pyramid is 1/4 of the size of previous level.
- The lowest level is of the highest resolution.
- The highest level is of the lowest resolution.

Pyramid

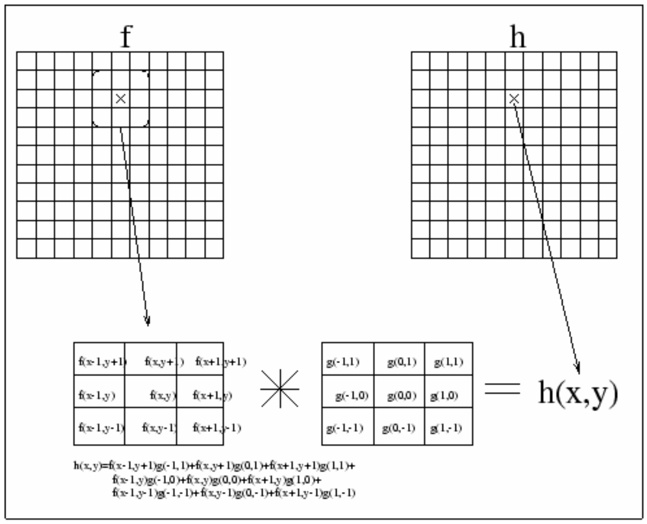


Gaussian Pyramids

$$g_l(i,j) = \sum_{m=-2}^{2}\sum_{n=-2}^{2} w(m,n)\, g_{l-1}(2i+m, 2j+n)$$

$$g_l = REDUCE[g_{l-1}]$$

Convolution



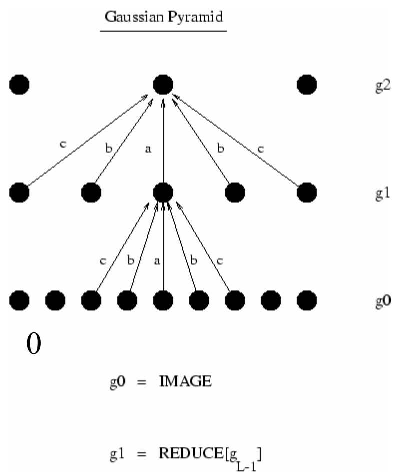
Reduce (1D)

$$g_l(i) = \sum_{m=-2}^{2} \hat{w}(m) g_{l-1}(2i+m)$$

$$g_l(2) = \hat{w}(-2) g_{l-1}(4-2) + \hat{w}(-1) g_{l-1}(4-1) + \\ \hat{w}(0) g_{l-1}(4) + \hat{w}(1) g_{l-1}(4+1) + \hat{w}(2) g_{l-1}(4+2)$$

$$g_l(2) = \hat{w}(-2) g_{l-1}(2) + \hat{w}(-1) g_{l-1} \hat{w}(3) + \\ \hat{w}(0) g_{l-1}(4) + \hat{w}(1) g_{l-1}(5) + \hat{w}(2) g_{l-1}(6)$$

Reduce



Gaussian Pyramids

$$g_{l,n}(i,j)=\sum_{p=-2}^{2}\sum_{q=-2}^{2}w(p,q)g_{l,n-1}\left(\frac{i-p}{2},\frac{j-q}{2}\right)$$

$$g_{l,n}=EXPAND[g_{l,n-1}]$$

Expand (1D)

$$g_{l,n}(i) = \sum_{p=-2}^{2} \hat{w}(p) g_{l,n-1}(\frac{i-p}{2})$$

$$g_{l,n}(4) = \hat{w}(-2) g_{l,n-1}(\frac{4-2}{2}) + \hat{w}(-1) g_{l,n-1}(\frac{4-1}{2}) +$$

$$\hat{w}(0) g_{l,n-1}(\frac{4}{2}) + \hat{w}(1) g_{l,n-1}(\frac{4+1}{2}) + \hat{w}(2) g_{l,n-1}(\frac{4+2}{2})$$

$$g_{l,n}(4) = \hat{w}(-2) g_{l,n-1}(1) + \hat{w}(0) g_{l,n-1}(2) + \hat{w}(2) g_{l,n-1}(3)$$

Expand (1D)

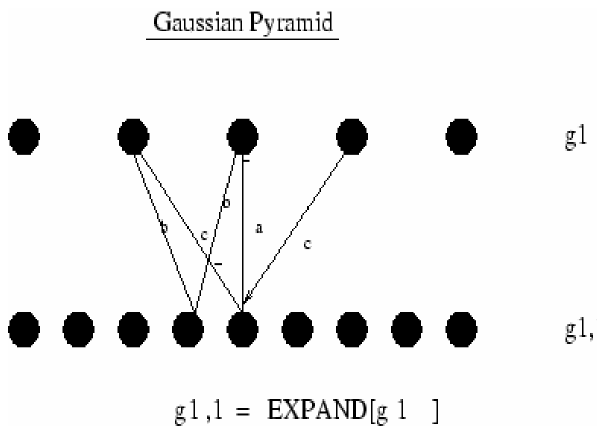
$$g_{l,n}(i) = \sum_{p=-2}^{2} \hat{w}(p) g_{l,n-1}(\frac{i-p}{2})$$

$$g_{l,n}(3) = \hat{w}(-2) g_{l,n-1}(\frac{3-2}{2}) + \hat{w}(-1) g_{l,n-1}(\frac{3-1}{2}) +$$

$$\hat{w}(0) g_{l,n-1}(\frac{3}{2}) + \hat{w}(1) g_{l,n-1}(\frac{3+1}{1}) + \hat{w}(2) g_{l,n-1}(\frac{3+2}{2})$$

$$g_{l,n}(3) = \hat{w}(-1) g_{l,n-1}(1) + \hat{w}(1) g_{l,n-1}(2)$$

Expand



Convolution Mask

$$[w(-2), w(-1), w(0), w(1), w(2)]$$

Convolution Mask

- Separable

$$w(m,n) = \hat{w}(m)\hat{w}(n)$$

•Symmetric

$$\hat{w}(i) = \hat{w}(-i)$$

$$[c,b,a,b,c]$$

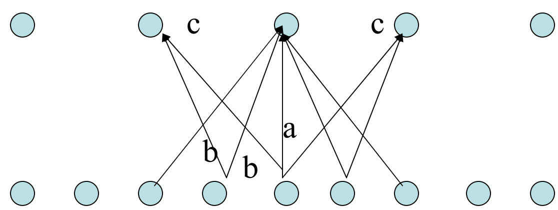
Convolution Mask

- The sum of mask should be 1.

$$a + 2b + 2c = 1$$

•All nodes at a given level must contribute the same total weight to the nodes at the next higher level.

$$a + 2c = 2b$$



Convolution Mask

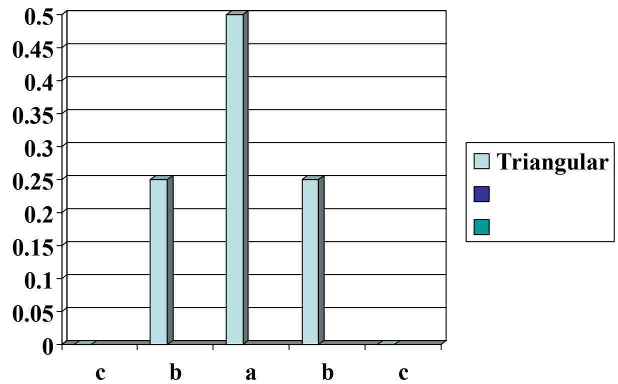
$$\hat{w}(0) = a$$

$$\hat{w}(-1) = \hat{w}(1) = \frac{1}{4}$$

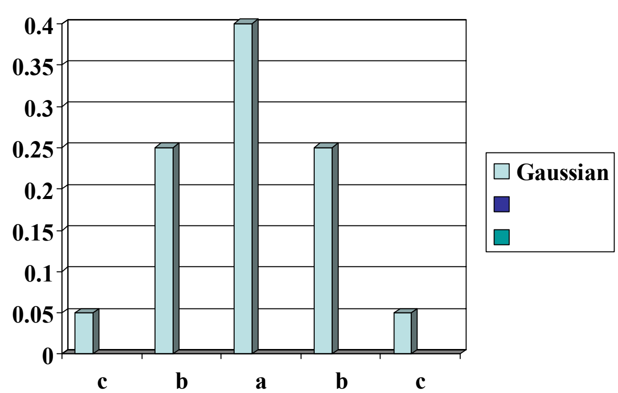
$$\hat{w}(-2) = \hat{w}(2) = \frac{1}{4} - \frac{a}{2}$$

a=.4 GAUSSIAN, a=.5 TRINGULAR

Triangular



Approximate Gaussian



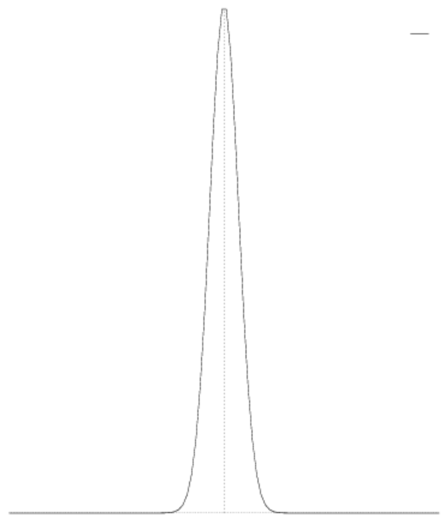
Gaussian

$$g(x) = e^{\frac{-x^2}{2o^2}}$$

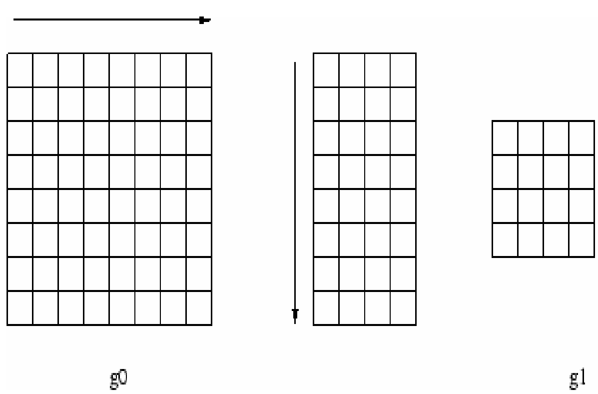
Gaussian

$$g(x) = e^{\frac{-x^2}{2o^2}}$$

x	-3	-2	-1	0	1	2	3
g(x)	.011	.13	.6	1	.6	.13	.011



Separability



Algorithm

- Apply 1-D mask to alternate pixels along each row of image.
- Apply 1-D mask to each pixel along alternate columns of resultant image from previous step.

Gaussian Pyramid

Laplacian Pyramids

- Similar to edge detected images.
- Most pixels are zero.
- Can be used for image compression.

$$L_1 = g_1 - EXPAND[g_2]$$

$$L_2 = g_2 - EXPAND[g_3]$$

$$L_3 = g_3 - EXPAND[g_4]$$

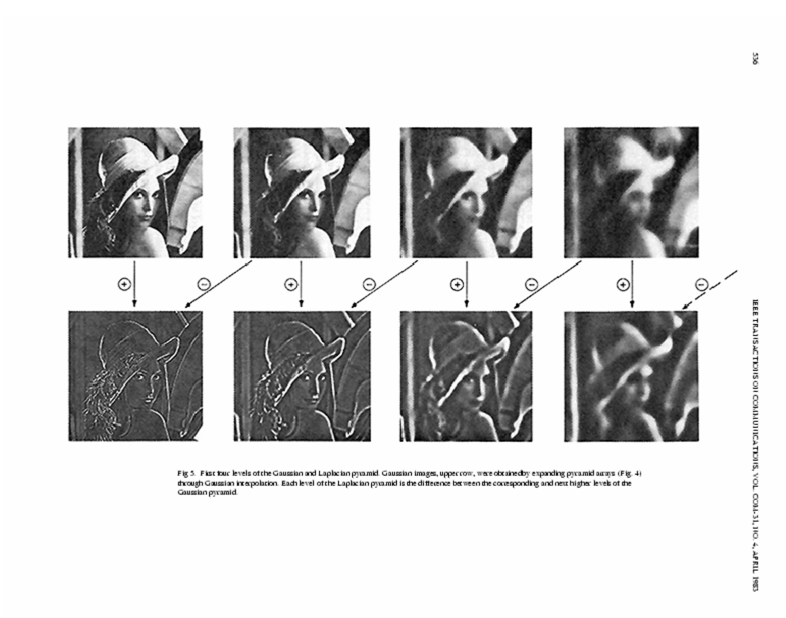


Fig. 5. First four levels of the Gaussian and Laplacian pyramid. Gaussian images, upper row, were obtained by expanding pyramid arrays (Fig. 4) through Gaussian interpolation. Each level of the Laplacian pyramid is the difference between the corresponding and next higher levels of the Gaussian pyramid.

Coding using Laplacian Pyramid

- Compute Gaussian pyramid

$$g_1, g_2, g_3, g_4$$

- Compute Laplacian pyramid

$$L_1 = g_1 - EXPAND[g_2]$$
$$L_2 = g_2 - EXPAND[g_3]$$
$$L_3 = g_3 - EXPAND[g_4]$$
$$L_4 = g_4$$

- Code Laplacian pyramid

Decoding using Laplacian pyramid

- Decode Laplacian pyramid.
- Compute Gaussian pyramid from Laplacian pyramid.

$$g_4 = L_4$$
$$g_3 = EXPAND[g_4] + L_3$$
$$g_2 = EXPAND[g_3] + L_2$$
$$g_1 = EXPAND[g_2] + L_1$$

- is reconstructed image.

Gaussian

- Most natural phenomenon can be modeled by Gaussian.
- Take a bunch of random variables of any distribution, find the mean, the mean will approach to Gaussian distribution.
- Gaussian is very smooth function, it has infinite no of derivatives.

Gaussian

- Fourier Transform of Gaussian is Gaussian.
- If you convolve Gaussian with itself, it is again Gaussian.
- There are cells in human brain which perform Gaussian filtering.
  - Laplacian of Gaussian edge detector

Carl F. Gauss

- Born to a peasant family in a small town in Germany.
- Learned counting before he could talk.
- Contributed to Physics, Mathematics, Astronomy,…
- Discovered most methods in modern mathematics, when he was a teenager.

Carl F. Gauss

- Some contributions
  - Gaussian elimination for solving linear systems
  - Gauss-Seidel method for solving sparse systems
  - Gaussian curvature
  - Gaussian quadrature

Laplacian Pyramid

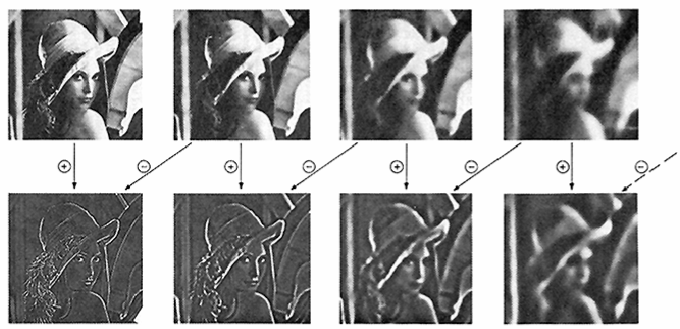


Fig 5. First four levels of the Gaussian and Laplacian pyramid. Gaussian images, upper row, were obtained by expanding pyramid arrays (Fig. 4) through Gaussian interpolation. Each level of the Laplacian pyramid is the difference between the corresponding and next higher levels of the Gaussian pyramid.

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Image Compression (Entropy)

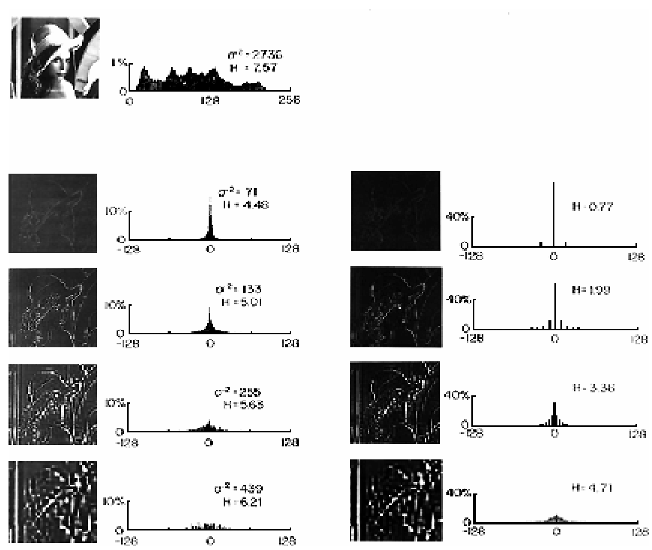
7.6

4.4 .77

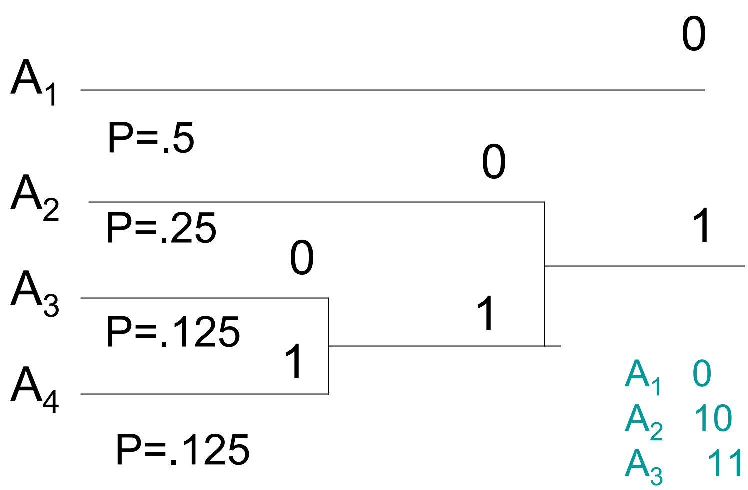
5.0 1.9

5.6 3.3

6.2 4.2



Huffman Coding (Example-1)



A_1 0
A_2 10
A_3 110
A_4 111

Huffman Coding

Entropy $$H = -\sum_{i=0}^{255} p(i)\log_2 p(i)$$

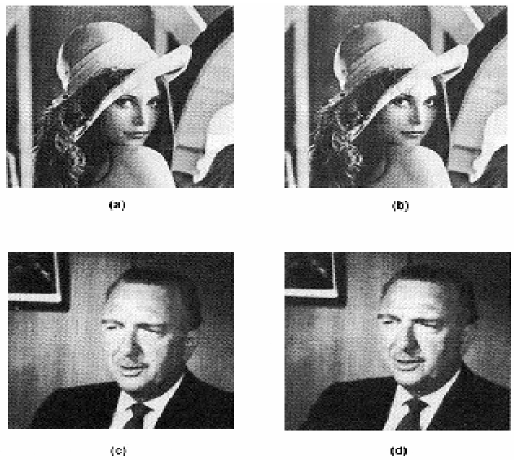
$$H = -.5\log .5 - .25\log .25 - .125\log .125 - .125\log .125 = 1.75$$

Image Compression

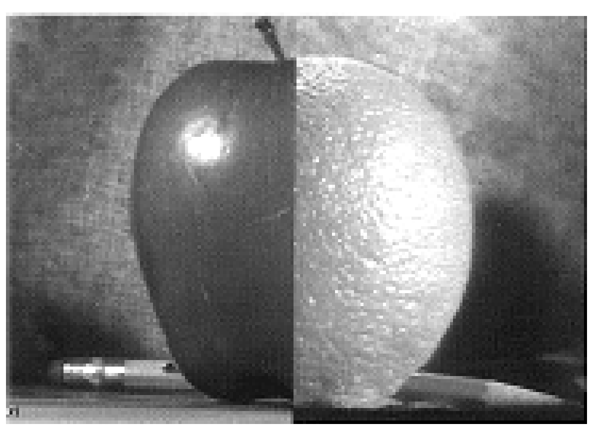
1.58

1

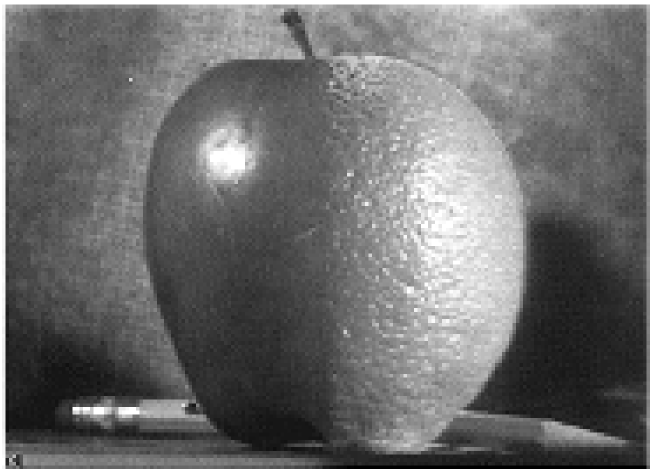
.73



Combining Apple & Orange



Combining Apple & Orange



Algorithm

- Generate Laplacian pyramid Lo of orange image.
- Generate Laplacian pyramid La of apple image.
- Generate Laplacian pyramid Lc by copying left half of nodes at each level from apple and right half of nodes from orange pyramids.
- Reconstruct combined image from Lc.

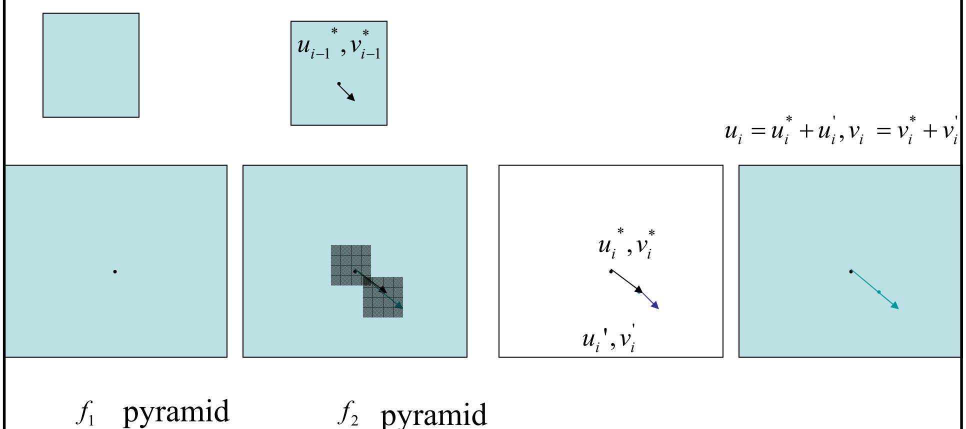
- http://ww-bcs.mit.edu/people/adelson/papers.html
  - The Laplacian Pyramid as a compact code, Burt and Adelson, IEEE Trans on Communication, 1983.

Lucas Kanade with Pyramids

- Compute 'simple' LK at highest level
- At level i
  - Take flow u_{i-1}, v_{i-1} from level i-1
  - bilinear interpolate it to create u_i^*, v_i^* matrices of twice resolution for level i
  - multiply u_i^*, v_i^* by 2
  - compute f_t from a block displaced by $u_i^*(x,y)$, $v_i^*(x,y)$
  - Apply LK to get $u_i'(x, y)$, $v_i'(x, y)$ (the correction in flow)
  - Add corrections u_i' v_i' , *i.e.* $u_i = u_i^* + u_i'$,

  $v_i = v_i^* + v_i'$.

Pyramids



Interpolation

	0	1	2	3
0	●	●	●	●
$u = 1$	●	●	●	●
2	●	●	●	●
3	●	●	●	●

	0	1	2	3
0	●	●	●	●
$v = 1$	●	●	●	●
2	●	●	●	●
3	●	●	●	●

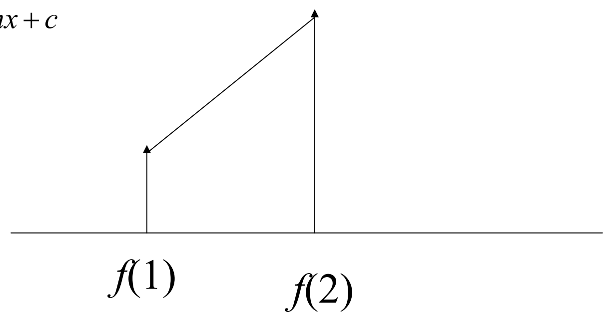
	0	1	2	3	4	5	6	7
0	●	○	●	○	●	○	●	○
1	○	○	○	○	○	○	○	○
2	●	○	●	○	●	○	●	○
$u^* = 3$	○	○	○	○	○	○	○	○
4	●	○	●	○	●	○	●	○
5	○	○	○	○	○	○	○	○
6	●	○	●	○	●	○	●	○
7	○	○	○	○	○	○	○	○

	0	1	2	3	4	5	6	7
0	●	○	●	○	●	○	●	○
1	○	○	○	○	○	○	○	○
2	●	○	●	○	●	○	●	○
$v^* = 3$	○	○	○	○	○	○	○	○
4	●	○	●	○	●	○	●	○
5	○	○	○	○	○	○	○	○
6	●	○	●	○	●	○	●	○
7	○	○	○	○	○	○	○	○

1-D Interpolation

$y = mx + c$

$f(x) = mx + c$



2-D Interpolation

$f(x,y) = a_1 + a_2 x + a_3 y + a_4 xy$

Bilinear

X X
O X

Bi-linear Interpolation

Four nearest points of (x,y) are:

$$(\underline{x},\underline{y}),(\overline{x},\underline{y}),(\underline{x},\overline{y}),(\overline{x},\overline{y})$$

$$(3,5),(4,5),(3,6),(4,6)$$

$\underline{x} = \text{int}(x)$ 3 (3.2,5.6)

$\underline{y} = \text{int}(y)$ 5

$\overline{x} = \underline{x} + 1$ 4

$\overline{y} = \underline{y} + 1$ 6

X (3,6) X (4,6)
X (3,5) X (4,5)

$$f'(x,y) = \overline{\varepsilon_x}\,\overline{\varepsilon_y} f(\underline{x},\underline{y}) + \underline{\varepsilon_x}\,\overline{\varepsilon_y} f(\overline{x},\underline{y}) +$$
$$\overline{\varepsilon_x}\,\underline{\varepsilon_y} f(\underline{x},\overline{y}) + \underline{\varepsilon_x}\,\underline{\varepsilon_y} f(\overline{x},\overline{y})$$

$\overline{\varepsilon_x} = \overline{x} - x$ $\overline{\varepsilon_x} = \overline{x} - x = 4 - 3.2 = .8$

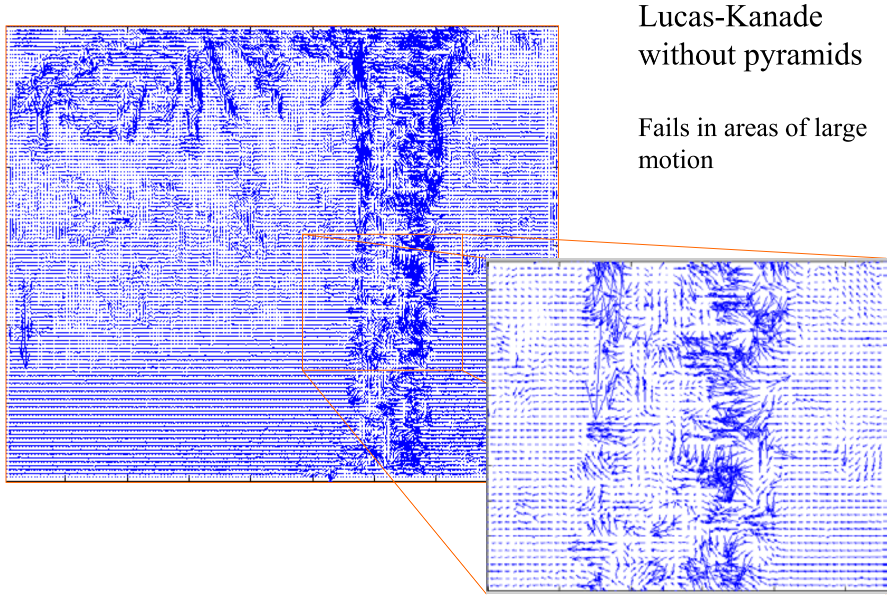
$\overline{\varepsilon_y} = \overline{y} - y$ $\overline{\varepsilon_y} = \overline{y} - y = 6 - 5.6 = .4$

$\underline{\varepsilon_x} = x - \underline{x}$ $\underline{\varepsilon_x} = x - \underline{x} = 3.2 - 2 = .2$

$\underline{\varepsilon_y} = y - \underline{y}$ $\underline{\varepsilon_y} = y - \underline{y} = 5.6 - 5 = .6$

Lucas-Kanade without pyramids

Fails in areas of large motion



Lucas-Kanade with Pyramids