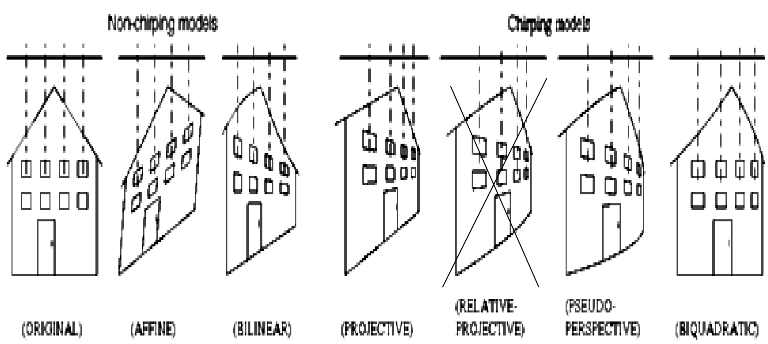
Lecture-3

Displacement Models (contd)



Affine Mosaic

Projective Mosaic

Instantaneous Velocity Model

3-D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} X'-X \\ Y'-Y \\ Z'-Z \end{bmatrix} = \begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \left(\begin{bmatrix} 0 & -\alpha & \beta \\ \alpha & 0 & -\gamma \\ -\beta & \gamma & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

3-D Rigid Motion

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \Omega = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix}$$

$$\dot{\mathbf{X}} = \Omega \times \mathbf{X} + \mathbf{V}$$

Cross Product

Orthographic Projection

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$y = Y$$
$$x = X$$

$$u = \dot{x} = \Omega_2 Z - \Omega_3 y + V_1$$
$$v = \dot{y} = \Omega_3 x - \Omega_1 Z + V_2$$

(u,v) is optical flow

Plane+orthographic(Affine)

$Z = a + bX + cY$

$$u = V_1 + \Omega_2 Z - \Omega_3 y$$
$$v = V_2 + \Omega_3 x - \Omega_1 Z$$

(1) Show this

$$u = b_1 + a_1 x + a_2 y$$
$$v = b_2 + a_3 x + a_4 y$$

$$\Downarrow$$

$$\mathbf{u} = \mathbf{A}\,\mathbf{x} + \mathbf{b}$$

$$b_1 = V_1 + a\Omega_2$$
$$a_1 = b\Omega_2$$
$$a_2 = c\Omega_2 - \Omega_3$$

$$b_2 = V_2 - a\Omega_1$$
$$a_3 = \Omega_3 - b\Omega_1$$
$$a_4 = -c\Omega_1$$

Perspective Projection (arbitrary flow)

$$x = \frac{fX}{Z} \qquad u = \dot{x} = \frac{fZ\dot{X} - fX\dot{Z}}{Z^2} = f\frac{\dot{X}}{Z} - x\frac{\dot{Z}}{Z}$$

$$y = \frac{fY}{Z} \qquad v = \dot{y} = \frac{fZ\dot{Y} - fY\dot{Z}}{Z^2} = f\frac{\dot{Y}}{Z} - y\frac{\dot{Z}}{Z}$$

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$u = f\left(\frac{V_1}{Z} + \Omega_2\right) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$

$$v = f\left(\frac{V_2}{Z} - \Omega_1\right) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

Plane+Perspective (pseudo perspective)

$$u = f\left(\frac{V_1}{Z} + \Omega_2\right) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$

$$v = f\left(\frac{V_2}{Z} - \Omega_1\right) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

$$Z = a + bX + cY$$

$$\frac{1}{Z} = \frac{1}{a} - \frac{b}{a}x - \frac{c}{a}y$$

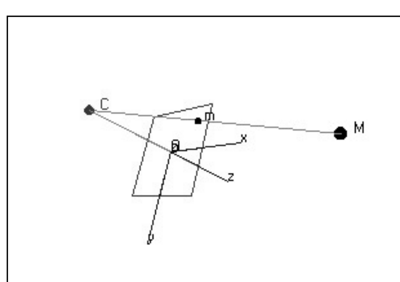
(2) Show this

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$$

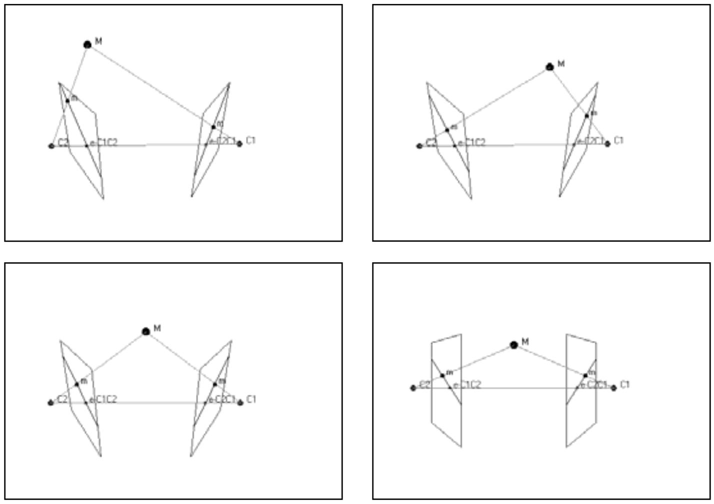
$$v = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2$$

3D Scenes

- When the scene is neither planar not distant, the relation between two consecutive frames is described by the *fundamental matrix*.
- What the heck is epipolar geometry?



Epi-polar Geometry



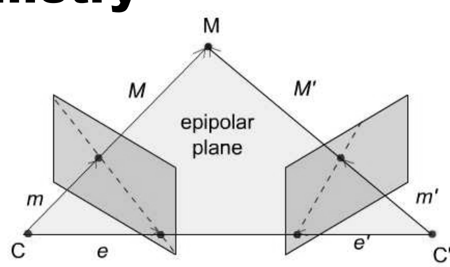
Epipolar Geometry

Epi-polar Geometry

- *Coplanar Condition*

  $(\boldsymbol{B} - \boldsymbol{A}) \cdot \boldsymbol{A} \times \boldsymbol{B} = 0$

- $\boldsymbol{M}' = \boldsymbol{R}\,(\boldsymbol{M} - \boldsymbol{T})$
- $(\boldsymbol{M} - \boldsymbol{T})^{\mathrm{T}} . \boldsymbol{T} \times \boldsymbol{M} = 0$
- $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$
- $(\boldsymbol{R}^{\mathrm{T}} \boldsymbol{M}')^{\mathrm{T}} . \boldsymbol{T} \times \boldsymbol{M} = 0$
- $\boldsymbol{A} \times \boldsymbol{B} = S \cdot \boldsymbol{B}$ *therefore* $\boldsymbol{M}'^{\mathrm{T}} \boldsymbol{R}\, S\, \boldsymbol{M} = 0$
- $\boldsymbol{M}'^{T} \boldsymbol{E}\, \boldsymbol{M} = 0$ *where* $\boldsymbol{E} = \boldsymbol{RS}$ *is the essential matrix*



Slide 14

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a1 The Essential Matrix is the mapping between points and epipolar lines
subail, 10/9/2001

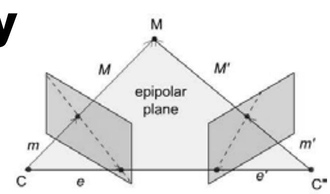
Vector Cross Product to Matrix-vector multiplication

$$A \times B = SB$$

$$S = \begin{bmatrix} 0 & -A_z & A_y \\ A_z & 0 & -A_x \\ -A_y & A_x & 0 \end{bmatrix}$$

Epipolar Geometry

Epipolar Geometry



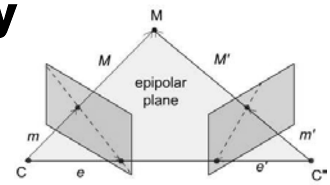
- The *Fundamental Matrix* is the *Essential Matrix* described in pixel coordinates.

- m and m′ are the points in pixel coordinates corresponding to **m** and **m**′

- $\boldsymbol{M} = \boldsymbol{P}^{-1}$ m and $\boldsymbol{M}' = \boldsymbol{P}'^{-1}$ m′

- $\boldsymbol{M}^{T} \boldsymbol{E} \boldsymbol{M} = 0$
- $\boldsymbol{m}^{T} \boldsymbol{E} \boldsymbol{m} = 0$
- $(\boldsymbol{P}'^{-1}\mathrm{m}')^{T} \boldsymbol{E} (\boldsymbol{P}^{-1}\mathrm{m}) = m^{T} \boldsymbol{P}'^{-T} \boldsymbol{E} \boldsymbol{P}^{-1}\mathrm{m}$

Epipolar Geometry



- The *Fundamental Matrix* is the *Essential Matrix* described in pixel coordinates.
- $(P'^{-1} m')^T E (P^{-1} m) = m'^T P'^{-T} E P^{-1} m = 0$
- $m'^T F m = 0$ where F is the fundamental matrix
- $F = P'^{-T} E P^{-1}$ The Ps are the Intrinsic parameters
- The Fundamental Matrix maps a point to the it's epipolar line in pixel coordinates

Fundamental Matrix

- Fundamental matrix captures the relationship between the corresponding points in two views.

$$m^T F m' = 0,$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} x'_i \\ y'_i \\ 1 \end{bmatrix} = 0,$$

$$\begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}^T \begin{bmatrix} f_{11}x' + f_{12}y'_i + f_{13} \\ f_{21}x' + f_{22}y'_i + f_{23} \\ f_{31}x' + f_{32}y'_i + f_{33} \end{bmatrix} = 0,$$

$$x_i(f_{11}x' + f_{12}y'_i + f_{13}) + y_i(f_{21}x' + f_{22}y'_i + f_{23}) + (f_{31}x' + f_{32}y'_i + f_{33}) = 0$$

Fundamental Matrix

$$x_i(f_{11}x' + f_{12}y'_i + f_{13}) + y_i(f_{21}x' + f_{22}y'_i + f_{23}) + (f_{31}x' + f_{32}y'_i + f_{33}) = 0$$

$$x_i x' f_{11} + x_i y'_i f_{12} + x_i f_{13} + y_i x' f_{21} + x' y'_i f_{22} + y'_i f_{23} + x' f_{31} + y'_i f_{32} + f_{33} = 0$$

$$Mf = \begin{bmatrix} x'_1 x_1 & x'_1 y_1 & x'_1 & y'_1 x_1 & y'_1 y_1 & y'_1 & x_1 & y_1 & 1 \\ x'_2 x_2 & x'_2 y_2 & x'_2 & y'_2 x_2 & y'_2 y_2 & y'_2 & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x'_n x_n & x'_n y_n & x'_1 & y'_n x_n & y'_n y_n & y'_n & x_n & y_n & 1 \end{bmatrix} f = 0$$

M is 9 by n matrix $\quad f = [f_{11} \quad f_{12} \quad f_{13} \quad f_{21} \quad f_{22} \quad f_{23} \quad f_{31} \quad f_{32} \quad f_{33}]$

To solve the equation, the rank(M) must be 8.

Computation of Fundamental Matrix

Normalized 8-point algorithm

- Objective:
  - Compute fundamental matrix F such that $\mathbf{x}_i'F\mathbf{x}_i = 0$
- Algorithm
  - Normalize the image by $\hat{\mathbf{x}}_i = T\mathbf{x}_i$ and $\hat{\mathbf{x}}_i' = T'\mathbf{x}_i'$
    - Find centroid of points in each image, determine the range, and normalize all points between 0 and 1
  - Linear solution
    - determining the eigen vector corresponding to the smallest eigen value of A,

$$Af = \begin{bmatrix} \hat{x}_1'\hat{x}_1 & \hat{x}_1'\hat{y}_1 & \hat{x}_1' & \hat{y}_1'\hat{x}_1 & \hat{y}_1'\hat{y}_1 & \hat{y}_1' & \hat{x}_1 & \hat{y}_1 & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \hat{x}_8'\hat{x}_8 & \hat{x}_8'\hat{y}_8 & \hat{x}_8' & \hat{y}_8'\hat{x}_8 & \hat{y}_8'\hat{y}_8 & \hat{y}_8' & \hat{x}_8' & \hat{y}_8 & 1 \end{bmatrix} f = 0$$

  - Construct $\hat{F} = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{bmatrix}$
  - Normalize

$$\hat{F} = \hat{F} / \| \hat{F} \|$$

Normalized 8-point algorithm

- Constraint enforcement
  - SVD decomposition

$$\hat{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} V' \qquad (\sigma_1 \geq \sigma_2 \geq \sigma_3)$$

  - Rank enforcement

$$\tilde{F} = U \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} V' \qquad (\sigma_3 = 0)$$

- De-normalization:

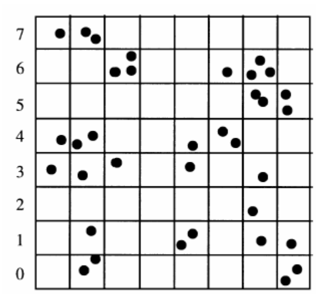
$$F = T^{-1^T} \tilde{F} T$$

(3) Show this

$$T = \begin{bmatrix} a_x & 0 & d_x \\ 0 & a_y & d_y \\ 0 & 0 & 1 \end{bmatrix}$$

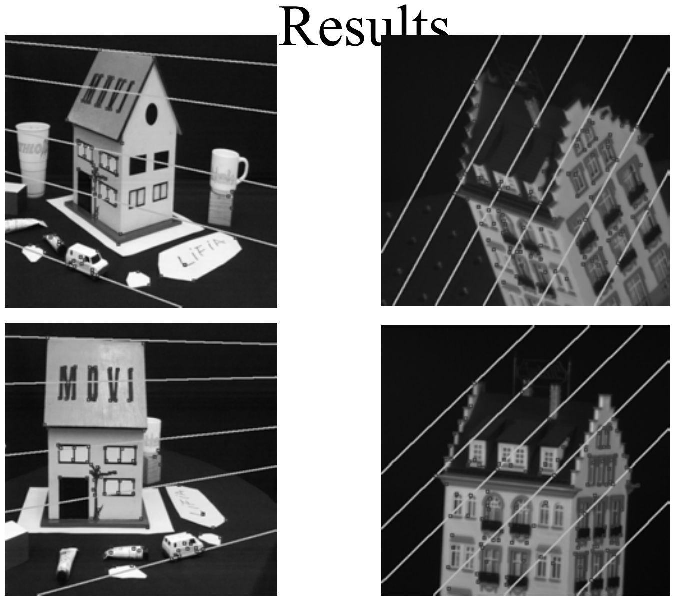
Robust Fundamental Matrix Estimation (by Zhang)

- Uniformly divide the image into 8×8 grid.
- Randomly select 8 grid cells and pick one pair of corresponding points from each grid cell, then use Hartley's 8-point algorithm to compute Fundamental Matrix F_i.
- For each F_i, compute the median of the squared residuals R_i.
  - $R_i = \text{median}_k[d(p_{1k}, F_i p_{2k}) + d(p_{2k}, F_i p_{1k})]$
- Select the best F_i according to R_i.
- Remove outlier correspondences if $R_k > Th$.
- Using the remaining points compute the fundamental Matrix F by weighted least square method.



Results

Results



Homework Due 9/16/2004

- Show (1), (2) and (3).