Lecture-5

Computing Optical Flow: Lucas & Kanade

Global Flow

Lucas & Kanade (Least Squares)

- Optical flow eq

$$f_x u + f_y v = -f_t$$

- Consider 3 by 3 window

$$f_{x1} u + f_{y1} v = -f_{t1}$$

$$\vdots$$

$$f_{x9} u + f_{y9} v = -f_{t9}$$

$$\begin{bmatrix} f_{x1} & f_{y1} \\ \vdots & \vdots \\ f_{x9} & f_{y9} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -f_{t1} \\ \vdots \\ -f_{t9} \end{bmatrix}$$

$$\mathbf{Au} = \mathbf{f_t}$$

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$$\mathbf{Au} = \mathbf{f_t}$$

$$\mathbf{A^T A u} = \mathbf{A^T f_t}$$

$$\mathbf{u} = (\mathbf{A^T A})^{-1} \mathbf{A^T f_t}$$

$$\min \sum (f_{xi} u + f_{yi} v + f_t)^2$$

Lucas & Kanade

$$\min \sum (f_{xi} u + f_{yi} v + f_t)^2$$

$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{xi} = 0$$

$$\sum (f_{xi} u + f_{yi} v + f_{ti}) f_{yi} = 0$$

$$\sum (f_{xi}u + f_{yi}v + f_{ti})f_{xi} = 0$$

$$\sum (f_{xi}u + f_{yi}v + f_{ti})f_{yi} = 0$$

$$\sum f_{xi}^2 u + \sum f_{xi}f_{yi}v = -\sum f_{xi}f_{ti}$$

$$\sum f_{xi}f_{yi}u + \sum f_{yi}^2 v = -\sum f_{yi}f_{ti}$$

$$\begin{bmatrix} \sum f_{xi}^2 & \sum f_{xi}f_{yi} \\ \sum f_{xi}f_{yi} & \sum f_{yi}^2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -\sum f_{xi}f_{ti} \\ -\sum f_{yi}f_{ti} \end{bmatrix}$$

Lucas & Kanade

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \sum f_{xi}^2 & \sum f_{xi}f_{yi} \\ \sum f_{xi}f_{yi} & \sum f_{yi}^2 \end{bmatrix}^{-1} \begin{bmatrix} -\sum f_{xi}f_{ti} \\ -\sum f_{yi}f_{ti} \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi}f_{yi})^2} \begin{bmatrix} \sum f_{yi}^2 & -\sum f_{xi}f_{yi} \\ -\sum f_{xi}f_{yi} & \sum f_{xi}^2 \end{bmatrix} \begin{bmatrix} -\sum f_{xi}f_{ti} \\ -\sum f_{yi}f_{ti} \end{bmatrix}$$

$$u = \frac{-\sum f_{yi}^2 \sum f_{xi}f_{ti} + \sum f_{xi}f_{yi} \sum f_{yi}f_{ti}}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi}f_{yi})^2}$$

$$v = \frac{\sum f_{xi}f_{ti} \sum f_{xi}f_{yi} - \sum f_{xi}^2 \sum f_{yi}f_{ti}}{\sum f_{xi}^2 \sum f_{yi}^2 - (\sum f_{xi}f_{yi})^2}$$

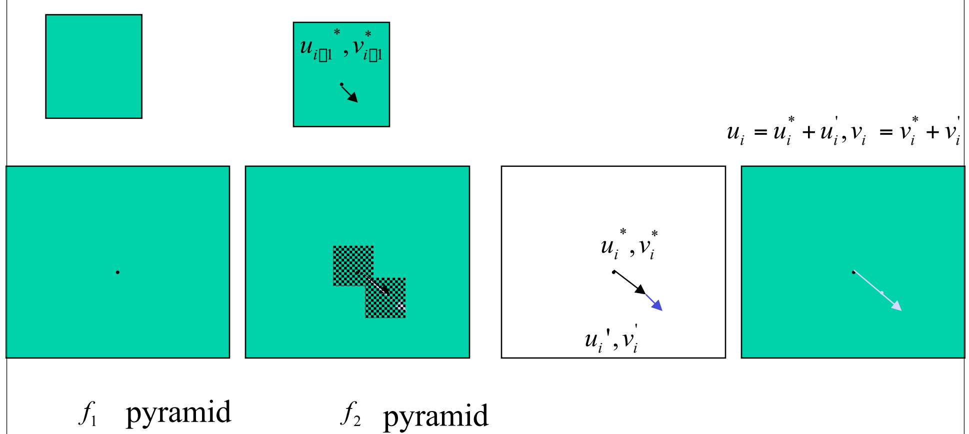
Comments

- Horn-Schunck and Lucas-Kanade optical method works only for small motion.
- If object moves faster, the brightness changes rapidly, 2x2 or 3x3 masks fail to estimate spatiotemporal derivatives.
- Pyramids can be used to compute large optical flow vectors.

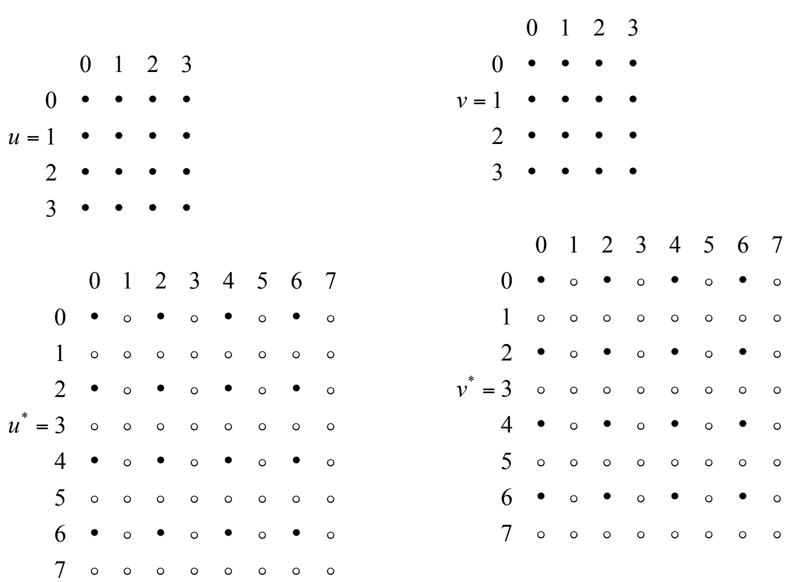
Lucas Kanade with Pyramids

- Compute 'simple' LK at highest level
- At level i
  - Take flow u_{i-1}, v_{i-1} from level i-1
  - bilinear interpolate it to create u_i^*, v_i^* matrices of twice resolution for level i
  - multiply u_i^*, v_i^* by 2
  - compute f_t from a block displaced by $u_i^*(x,y)$, $v_i^*(x,y)$
  - Apply LK to get $u_i'(x, y)$, $v_i'(x, y)$ (the correction in flow)
  - Add corrections u_i' v_i', *i.e.* $u_i = u_i^* + u_i'$, $v_i = v_i^* + v_i'$.

Pyramids

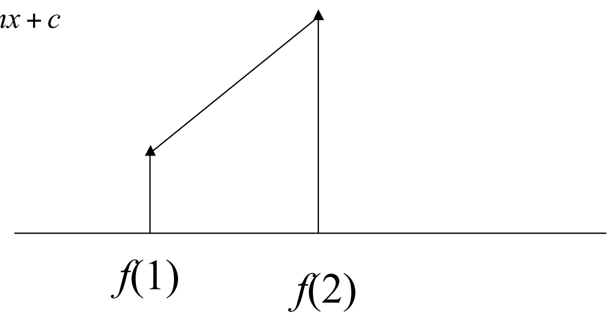


Interpolation



1-D Interpolation

$$y = mx + c$$
$$f(x) = mx + c$$



2-D Interpolation

$$f(x, y) = a_1 + a_2 x + a_3 y + a_4 xy$$

Bilinear

X X
O X

Bi-linear Interpolation

Four nearest points of (x,y) are:

$$(\underline{x},\underline{y}),(\bar{x},\underline{y}),(\underline{x},\bar{y}),(\bar{x},\bar{y})$$

$$(3,5),(4,5),(3,6),(4,6)$$

$\underline{x} = \text{int}(x)$ 3 (3.2,5.6)

$\underline{y} = \text{int}(y)$ 5

$\bar{x} = \underline{x} + 1$ 4

$\bar{y} = \underline{y} + 1$ 6

X (3,6) X (4,6)

X (3,5) X (4,5)

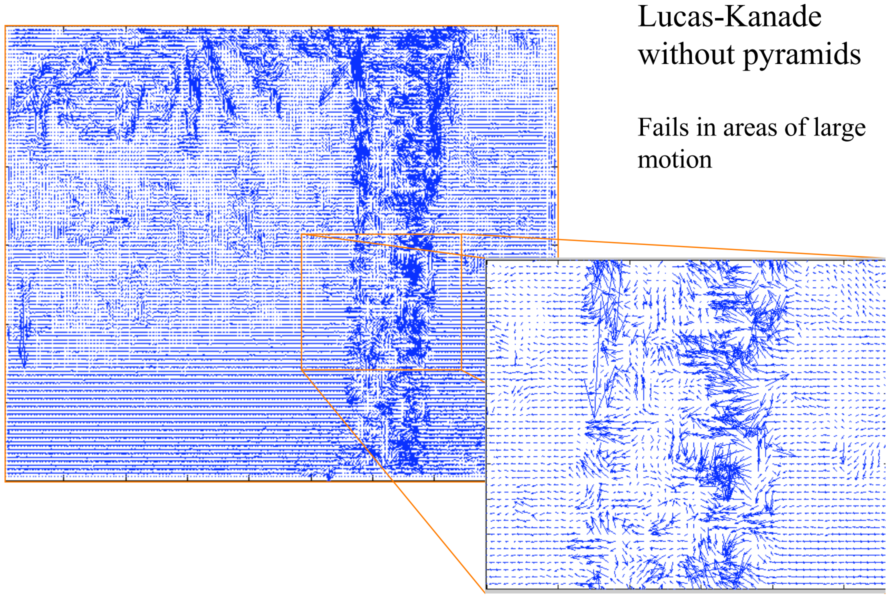
$$f\square(x,y) = \overline{\square_x}\,\overline{\square_y} f(\underline{x},\underline{y}) + \underline{\square_x}\,\overline{\square_y} f(\bar{x},\underline{y}) + \overline{\square_x}\,\underline{\square_y} f(\underline{x},\bar{y}) + \underline{\square_x}\,\underline{\square_y} f(\bar{x},\bar{y})$$

$\overline{\square_x} = \bar{x} - x$ $\overline{\square_x} = \bar{x} - x = 4 - 3.2 = .8$

$\overline{\square_y} = \bar{y} - y$ $\overline{\square_y} = \bar{y} - y = 6 - 5.6 = .4$

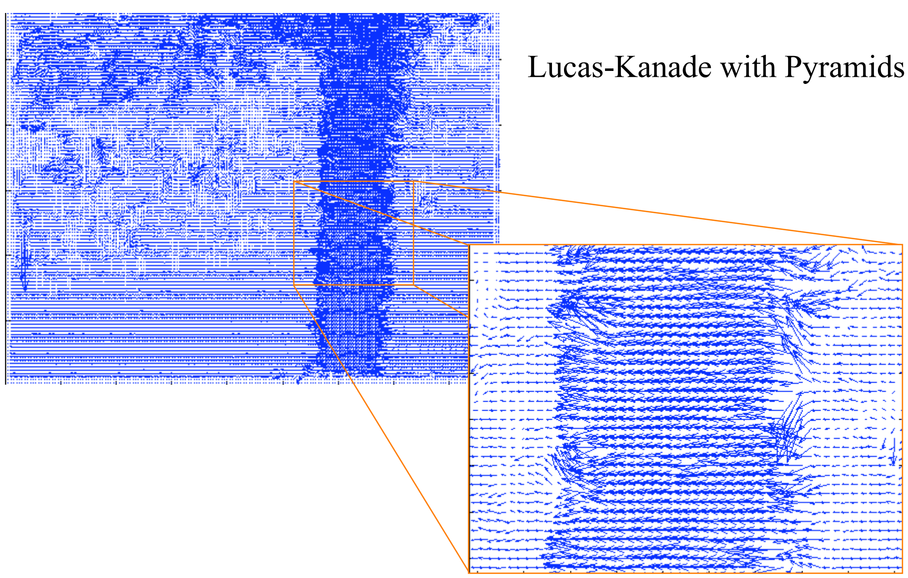
$\underline{\square_x} = x - \underline{x}$ $\underline{\square_x} = x - \underline{x} = 3.2 - 2 = .2$

$\underline{\square_y} = y - \underline{y}$ $\underline{\square_y} = y - \underline{y} = 5.6 - 5 = .6$

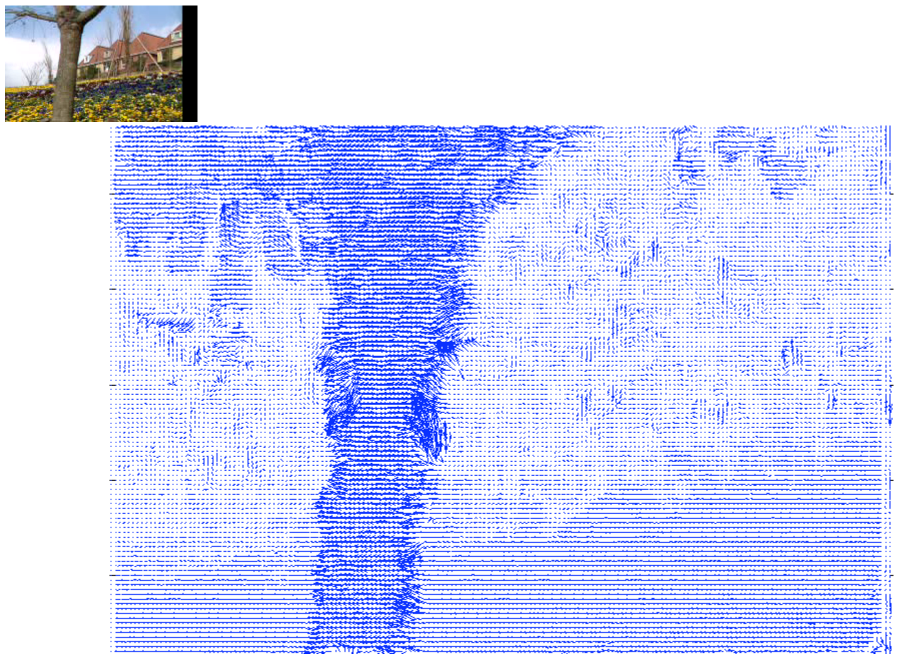


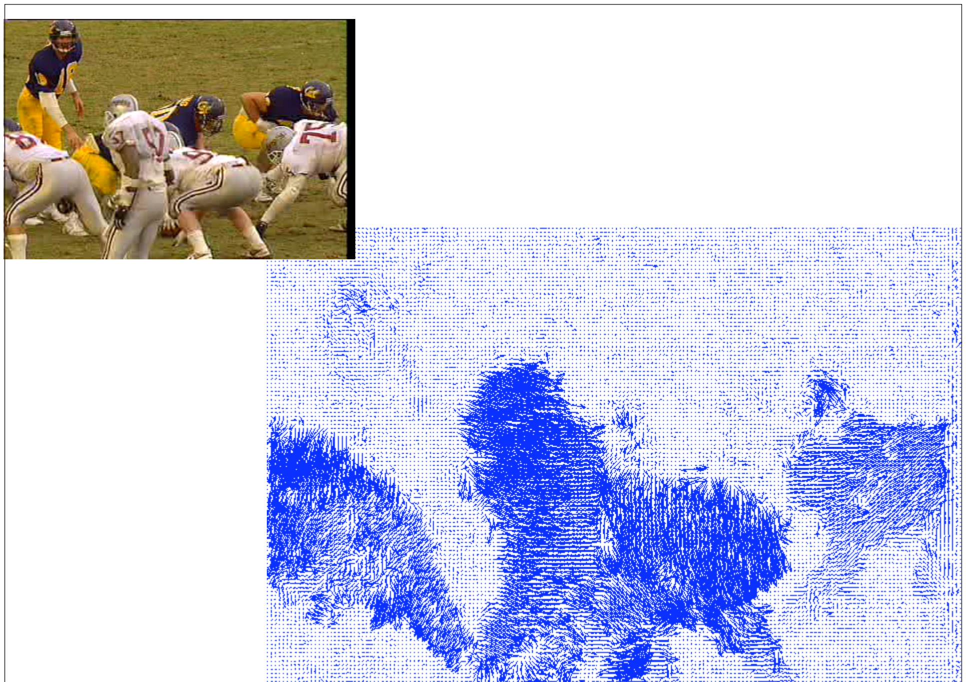
Lucas-Kanade without pyramids

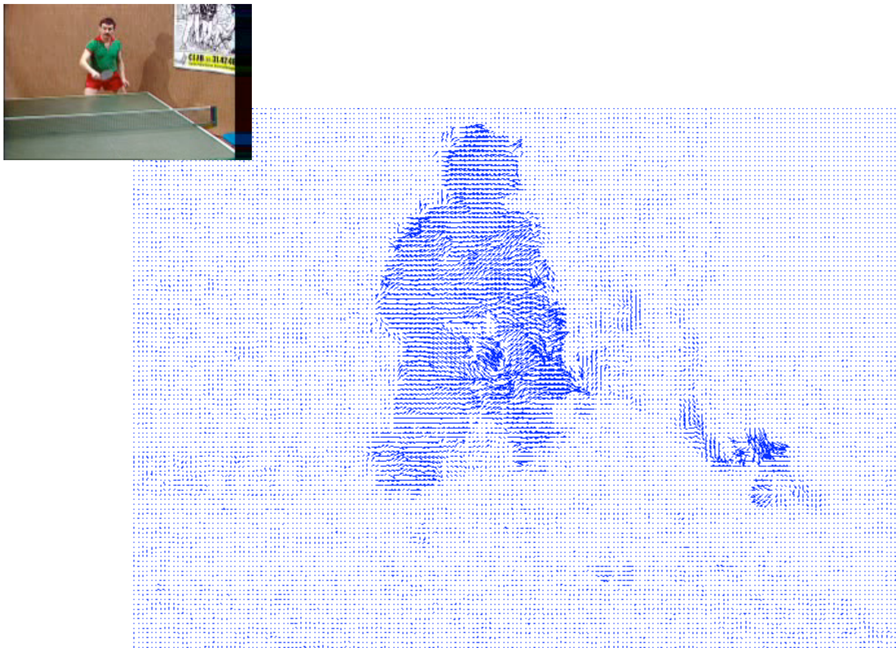
Fails in areas of large motion



Lucas-Kanade with Pyramids







Global Flow

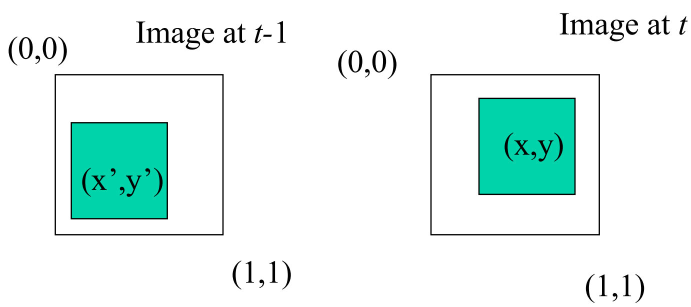
Anandan

Affine

Global Motion

- Estimate motion using all pixels in the image.
- Parametric flow gives an equation, which describes optical flow for each pixel.
  - Affine
  - Projective
- Global motion can be used to
  - generate mosaics
  - Object-based segmentation

Affine



$$u(x,y) = a_1 x + a_2 y + b_1$$
$$v(x,y) = a_3 x + a_4 y + b_2$$

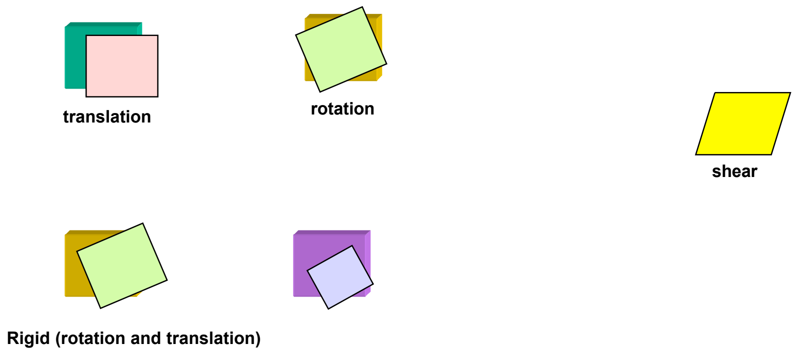
$$X' = X - U$$

Affine

$$u(x,y) = a_1 x + a_2 y + b_1$$
$$v(x,y) = a_3 x + a_4 y + b_2$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Spatial Transformations



Anandan

$$u(x,y) = a_1 x + a_2 y + b_1$$
$$v(x,y) = a_3 x + a_4 y + b_2$$

•Affine

$$\begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix} = \begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ a_3 \\ a_4 \\ b_2 \end{bmatrix}$$

$$\mathbf{u}(\mathbf{x}) = \mathbf{X}(\mathbf{x})\mathbf{a}$$

Anandan

$$\mathbf{u}(\mathbf{x}) = \mathbf{X}(\mathbf{x})\mathbf{a}$$

Optical flow constraint eq $f_x u + f_y v = -f_t$

$$E(\delta a) = \sum_{x \in f(x,y)} (f_t + f_{\mathbf{x}}^T \delta u)^2 \qquad f_{\mathbf{x}} = \begin{bmatrix} f_x \\ f_y \end{bmatrix}$$

$$E(\delta a) = \sum_{x \in f(x,y)} (f_t + f_{\mathbf{x}}^T \mathbf{X} \delta a)^2$$

min

$$\left[\sum X^T (f_X)(f_x)^T X\right] \delta a = -\sum X^T f_X f_t$$

$$Ax = b$$

Linear system

Basic Components

- Pyramid construction
- Motion estimation
- Image warping
- Coarse-to-fine refinement

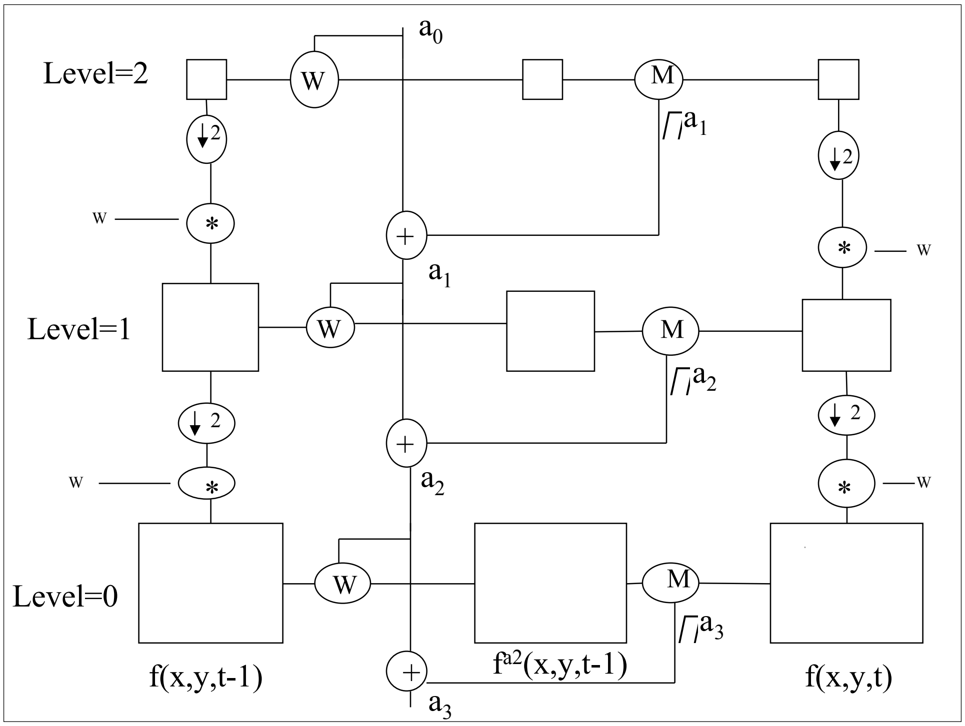


Image Warping

- Warping an image *f* into image *h* using some transformation *g*, involves mapping intensity at each pixel (*x,y*) in image *f* to a pixel (*g(x),* g(*y*)) image *h* such that

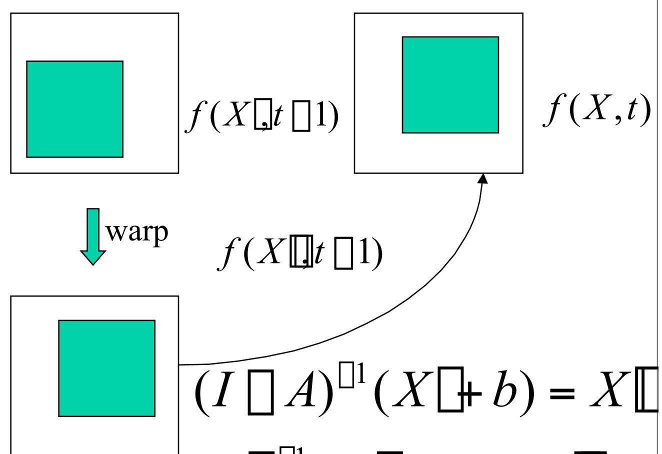
$$(x', y') = (g(x), g(y))$$

- In case of affine transformation, $\mathbf{x} = (x, y)$ is transformed to $\mathbf{x}' = (x', y')$ as:

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}$$

Image Warping

$$X' = X - U$$
$$= X - (AX + b)$$



$$(I - A)^{-1}(X' + b) = X''$$
$$(A')^{-1}(X' + b) = X''$$

Image Warping

$$X' = X - U = X - (AX + b)$$
$$X' = (I - A)X - b$$
$$X' = A'X - b$$
$$X' + b = A'X$$
$$(A')^{-1}(X' + b) = X$$

Image at time t: X
Image at time t-1: X'

$$(A')^{-1}(X' + b) = X'' \qquad X'' \approx X$$

Image Warping

- How about values in $X' = (x', y')$ are not integer.
- But image is sampled only at integer rows and columns
  - Instead of converting X' to X' and copying $f(X', t-1)$ at $f(X', t-1)$ we can convert integer values X' to X' and copy $f(X', t-1)$ at $f(X', t-1)$

Image Warping

- But how about the values in X' are not integer.

- Perform bilinear interpolation to compute $f(X', t-1)$ at non-integer values.

Image Warping

$$(A')^{-1}(X' + b) = X''$$

$$(X' + b) = (A')X''$$

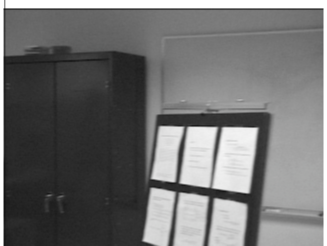
$$X' = (A')X'' - b \qquad X'' \approx X$$

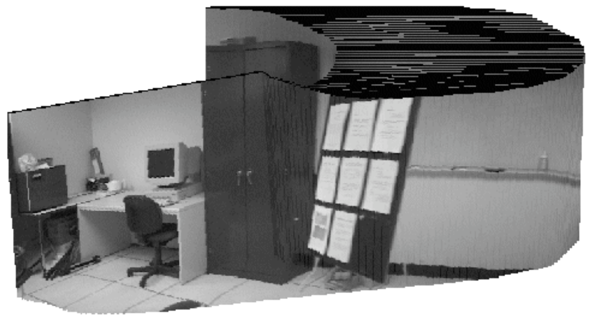
Warping

Show Demos

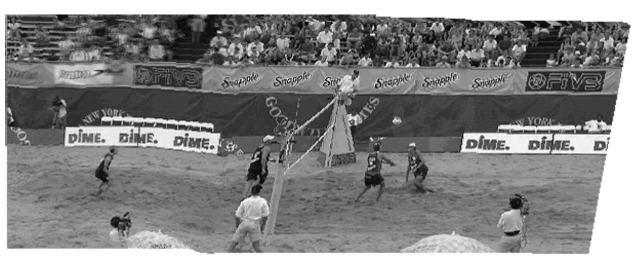
Video Mosaic

Video Mosaic





Video Mosaic



Sprite