PART I

Measurement of Motion

Contents

- Image Motion Models
- Optical Flow Methods
  - Horn & Schunck
  - Lucas and Kanade
  - Anandan et al
  - Szeliski
  - Mann & Picard
- Video Mosaics

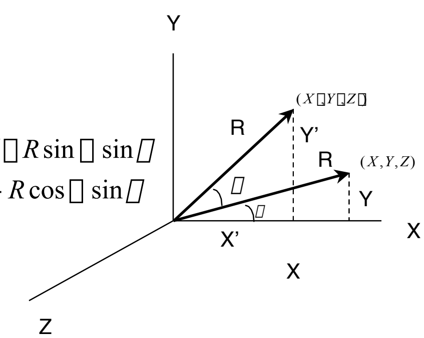
3-D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Rotation matrix (9 unknowns) Translation (3 unknowns)

Rotation

$$X = R\cos\phi$$
$$Y = R\sin\phi$$



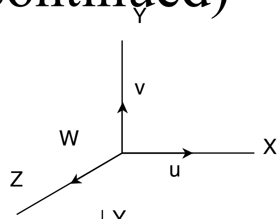
$$X' = R\cos(\theta+\phi) = R\cos\phi\cos\theta - R\sin\phi\sin\theta$$
$$Y' = R\sin(\theta+\phi) = R\sin\phi\cos\theta + R\cos\phi\sin\theta$$

$$X' = X\cos\theta - Y\sin\theta$$
$$Y' = X\sin\theta + Y\cos\theta$$

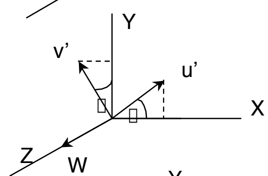
$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Rotation (continued)

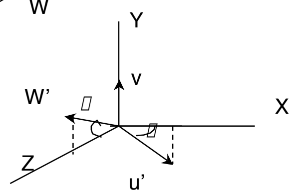
$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$



Euler Angles

$$R = R_Z^{\gamma} R_Y^{\beta} R_X^{\alpha} = \begin{bmatrix} \cos\gamma\cos\beta & \cos\gamma\sin\beta\sin\alpha - \sin\gamma\cos\alpha & \cos\gamma\sin\beta\cos\alpha + \sin\gamma\sin\alpha \\ \sin\gamma\cos\beta & \sin\gamma\sin\beta\sin\alpha + \cos\gamma\cos\alpha & \sin\gamma\sin\beta\cos\alpha - \cos\gamma\sin\alpha \\ -\sin\beta & \cos\beta\sin\alpha & \cos\beta\cos\alpha \end{bmatrix}$$

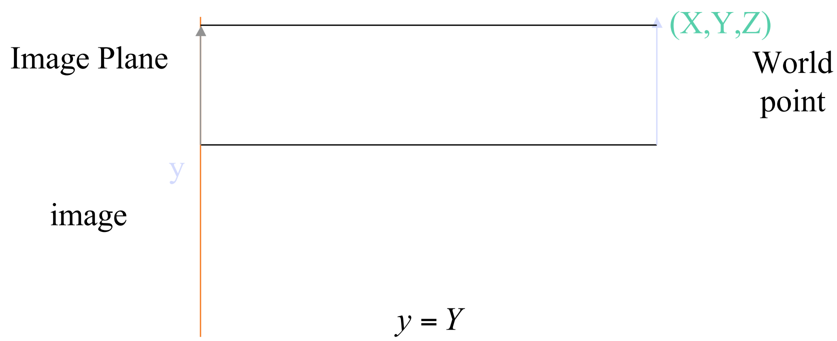
if angles are small($\cos\theta \approx 1 \quad \sin\theta \approx \theta$)

$$R = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}$$

Image Motion Models

Displacement Model

Orthographic Projection



$$y = Y$$
$$x = X$$

Orthographic Projection

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$x = X$$
$$y = Y$$

(x,y)=image coordinates, (X,Y,Z)=world coordinates

$$x' = r_{11}x + r_{12}y + (r_{13}Z + T_X)$$
$$y' = r_{21}x + r_{22}y + (r_{23}Z + T_Y)$$

$$x' = a_1 x + a_2 y + b_1$$
$$y' = a_3 x + a_4 y + b_2$$

Affine Transformation

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

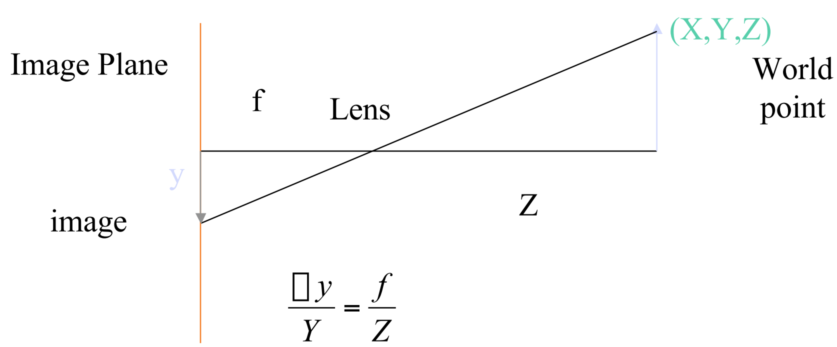
Orthographic Projection (contd.)

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = R\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + T = \begin{bmatrix} 1 & -\alpha & \beta \\ \alpha & 1 & -\gamma \\ -\beta & \gamma & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$x' = x - \alpha y + \beta Z + T_X$$

$$y' = \alpha x + y - \gamma Z + T_Y$$

Perspective Projection



$$\frac{-y}{Y} = \frac{f}{Z}$$

$$y = -\frac{fY}{Z} \qquad x = -\frac{fX}{Z}$$

Perspective Projection

$$\begin{bmatrix}X'\\Y'\\Z'\end{bmatrix} = R\begin{bmatrix}X\\Y\\Z\end{bmatrix} + T = \begin{bmatrix}r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix} + \begin{bmatrix}T_X\\T_Y\\T_Z\end{bmatrix}$$

$$X' = r_{11}X + r_{12}Y + r_{13}Z + T_X$$
$$Y' = r_{21}X + r_{22}Y + r_{23}Z + T_Y$$
$$Z' = r_{31}X + r_{32}Y + r_{33}Z + T_Z$$

$$x' = \frac{X'}{Z'} \quad y' = \frac{Y'}{Z'}$$

focal length = -1

$$x' = \frac{r_{11}x + r_{12}y + r_{13} + \frac{T_X}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}}$$

scale ambiguity

$$y' = \frac{r_{21}x + r_{22}y + r_{23} + \frac{T_Y}{Z}}{r_{31}x + r_{32}y + r_{33} + \frac{T_Z}{Z}}$$

Plane+Perspective(projective)

$$aX + bY + cZ = 1$$

$$\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix} = 1$$

equation of a plane

$$\begin{bmatrix}X'\\Y'\\Z'\end{bmatrix} = R\begin{bmatrix}X\\Y\\Z\end{bmatrix} + T$$

3d rigid motion

$$\begin{bmatrix}X'\\Y'\\Z'\end{bmatrix} = R\begin{bmatrix}X\\Y\\Z\end{bmatrix} + T\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}X\\Y\\Z\end{bmatrix}$$

Plane+Perspective(projective)

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = A \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

focal length = -1

$$X' = a_1 X + a_2 Y + a_3 Z$$
$$Y' = a_4 X + a_5 Y + a_6 Z$$
$$Z' = a_7 X + a_8 Y + a_9 Z$$

$$x' = \frac{a_1 X + a_2 Y + a_3 Z}{a_7 X + a_8 Y + a_9 Z}$$

$$y' = \frac{a_4 X + a_5 Y + a_6 Z}{a_7 X + a_8 Y + a_9 Z}$$

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + a_9}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + a_9}$$

$a_9 = 1$

scale ambiguity

Plane+perspective (contd.)

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}$$

$$\mathbf{X}' = \frac{\mathbf{A}\mathbf{X} + b}{C^T \mathbf{X} + 1}$$

$$\mathbf{X}' = \begin{bmatrix} x' \\ y' \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_4 & a_5 \end{bmatrix},$$

$$b = \begin{bmatrix} a_3 \\ a_6 \end{bmatrix}, C = \begin{bmatrix} a_7 \\ a_8 \end{bmatrix}, \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Projective

Least Squares

- Eq of a line

$$mx + c = y$$

  - Consider n points

$$mx_1 + c = y_1$$

$$\vdots$$

$$mx_n + c = y_n$$

$$\begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\mathbf{A}p = Y$$

Least Squares Fit

$$\mathbf{A}p = Y$$

$$\mathbf{A}^{\mathbf{T}}\mathbf{A}p = \mathbf{A}^{\mathbf{T}}Y$$

$$p = (\mathbf{A}^{\mathbf{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathbf{T}}Y$$

$$\min \sum_{i=1}^{n} (y_i - mx_i - c)^2$$

Projective

- If point correspondences (x,y)<-->(x',y') are known
- *a'*s can be determined by least squares fit

$$x' = \frac{a_1 x + a_2 y + a_3}{a_7 x + a_8 y + 1}$$

$$y' = \frac{a_4 x + a_5 y + a_6}{a_7 x + a_8 y + 1}$$

$$\begin{bmatrix} & & & & \vdots & & & \\ x_i & y_i & 1 & 0 & 0 & 0 & -x_i x_i' & -y_i x_i' \\ 0 & 0 & 0 & x_i & y_i & 1 & -x_i y_i' & -y_i y_i' \\ & & & & \vdots & & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix}$$

Affine

- If point correspondences (x,y)<-->(x',y') are known
- *a'*s and *b's* can be determined by least squares fit

$$x' = a_1 x + a_2 y + b_1$$
$$y' = a_3 x + a_4 y + b_2$$

$$\begin{bmatrix} & & & \vdots & & \\ x_i & y_i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_i & y_i & 1 \\ & & & \vdots & & \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ a_3 \\ a_4 \\ b_2 \end{bmatrix} = \begin{bmatrix} \vdots \\ x_i' \\ y_i' \\ \vdots \end{bmatrix}$$

Summary of Displacement Models

Translation

$$x' = x + b_1$$
$$y' = y + b_2$$

Biquadrati

$$x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y^2 + a_6 xy$$
$$y' = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} y^2 a_{12} xy$$

Rigid

$$x' = x\cos\theta - y\sin\theta + b_1$$
$$y' = x\sin\theta + y\cos\theta + b_2$$

Bilinear

$$x' = a_1 + a_2 x + a_3 y + a_4 xy$$
$$y' = a_5 + a_6 x + a_7 y + a_8 xy$$

Affine

$$x' = a_1 x + a_2 y + b_1$$
$$y' = a_3 x + a_4 y + b_2$$

Pseudo Perspective

$$x' = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$$
$$y' = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2$$

Projective

$$x' = \frac{a_1 x + a_2 y + b_1}{c_1 x + c_2 y + 1}$$
$$y' = \frac{a_3 x + a_4 y + b_1}{c_1 x + c_2 y + 1}$$

Displacement Models (contd)

- Translation
  - simple
  - used in block matching
  - no zoom, no rotation, no pan and tilt
- Rigid
  - rotation and translation
  - no zoom, no pan and tilt

Displacement Models (contd)

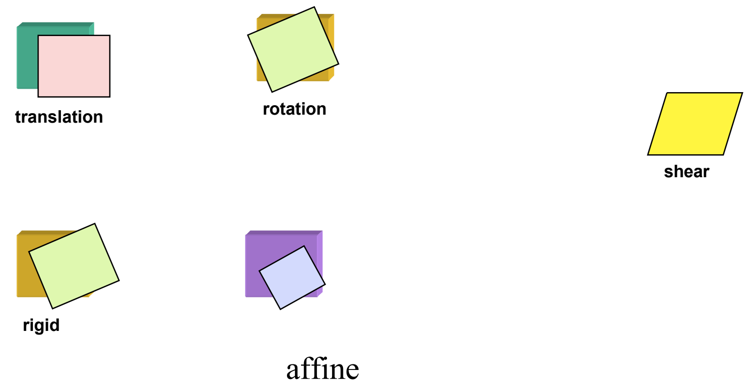
- Affine
  - rotation about optical axis only
  - can not capture pan and tilt
  - orthographic projection

- Projective
  - exact eight parameters (3 rotations, 3 translations and 2 scalings)
  - difficult to estimate

Displacement Models (contd)

- Biquadratic
  - obtained by second order Taylor series
  - 12 parameters
- Bilinear
  - obtained from biquadratic model by removing square terms
  - most widely used
  - not related to any physical 3D motion
- Pseudo-perspective
  - obtained by removing two square terms and constraining four remaining to 2 degrees of freedom

Spatial Transformations



Decomposition of Affine

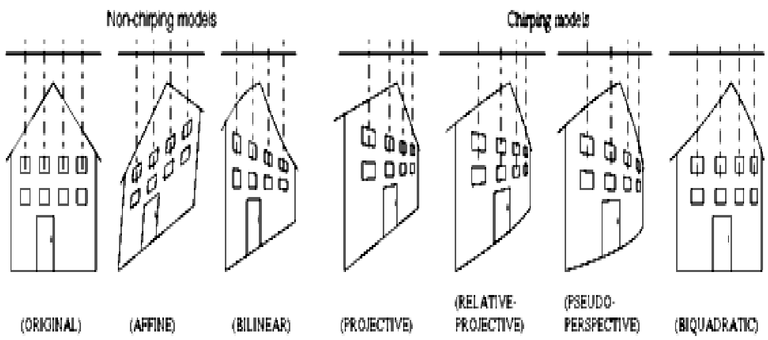
$$A = SVD = S(DD^{\Box 1})VD = (SD)(D^{\Box 1}VD)$$

$$= R(\Box)C = \begin{bmatrix} \cos\Box & \Box\sin\Box \\ \sin\Box & \cos\Box \end{bmatrix} \begin{bmatrix} v_1 & v_h \\ v_h & v_2 \end{bmatrix}$$

$$A = \Box \begin{bmatrix} \Box 1 & 0 \\ \Box\Box & \Box\Box \end{bmatrix} \begin{bmatrix} \cos\Box & \Box\sin\Box \\ \sin\Box & \cos\Box \end{bmatrix} \Box$$

$$\Box = scale\_factor = \sqrt{s_x s_y}, \quad \Box = scale\_ratio = \sqrt{\frac{s_x}{s_y}}, \quad \Box = \text{skew}$$

Displacement Models (contd)



Affine Mosaic

Projective Mosaic

Instantaneous Velocity Model

3-D Rigid Motion

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} X' - X \\ Y' - Y \\ Z' - Z \end{bmatrix} = \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \left(\begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

3-D Rigid Motion

$$\dot{X} = \omega_2 Z - \omega_3 Y + V_1$$
$$\dot{Y} = \omega_3 X - \omega_1 Z + V_2$$
$$\dot{Z} = \omega_1 Y - \omega_2 X + V_3$$

$$\mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \Omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

$$\dot{\mathbf{X}} = \Omega \times \mathbf{X} + \mathbf{V}$$

Cross Product

Orthographic Projection

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$y = Y$$
$$x = X$$

$$u = \dot{x} = \Omega_2 Z - \Omega_3 y + V_1$$
$$v = \dot{y} = \Omega_3 x - \Omega_1 Z + V_2$$

(u,v) is optical flow

Perspective Projection (arbitrary flow)

$$x = \frac{fX}{Z} \qquad u = \dot{x} = \frac{fZ\dot{X} - fX\dot{Z}}{Z^2} = f\frac{\dot{X}}{Z} - x\frac{\dot{Z}}{Z}$$
$$y = \frac{fY}{Z} \qquad v = \dot{y} = \frac{fZ\dot{Y} - fY\dot{Z}}{Z^2} = f\frac{\dot{Y}}{Z} - y\frac{\dot{Z}}{Z}$$

$$\dot{X} = \Omega_2 Z - \Omega_3 Y + V_1$$
$$\dot{Y} = \Omega_3 X - \Omega_1 Z + V_2$$
$$\dot{Z} = \Omega_1 Y - \Omega_2 X + V_3$$

$$u = f(\frac{V_1}{Z} + \Omega_2) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$
$$v = f(\frac{V_2}{Z} - \Omega_1) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

Plane+orthographic(Affine)

$$Z = a + bX + cY$$

$$u = V_1 + \Omega_2 Z - \Omega_3 y$$
$$v = V_2 + \Omega_3 x - \Omega_1 Z$$

$$u = b_1 + a_1 x + a_2 y$$
$$v = b_2 + a_3 x + a_4 y$$

$$b_1 = V_1 + a\Omega_2$$
$$a_1 = b\Omega_2$$
$$a_2 = c\Omega_2 - \Omega_3$$

$$b_2 = V_2 - a\Omega_1$$
$$a_3 = \Omega_3 - b\Omega_1$$
$$a_4 = -c\Omega_1$$

Plane+Perspective (pseudo perspective)

$$u = f\left(\frac{V_1}{Z} + \Omega_2\right) - \frac{V_3}{Z}x - \Omega_3 y - \frac{\Omega_1}{f}xy + \frac{\Omega_2}{f}x^2$$
$$v = f\left(\frac{V_2}{Z} - \Omega_1\right) + \Omega_3 x - \frac{V_3}{Z}y + \frac{\Omega_2}{f}xy - \frac{\Omega_1}{f}y^2$$

$$Z = a + bX + cY$$
$$\frac{1}{Z} = \frac{1}{a} - \frac{b}{a}x - \frac{c}{a}y$$

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy$$
$$v = a_6 + a_7 x + a_8 y + a_4 xy + a_5 y^2$$