On Satisfactory Partitioning of Graphs

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This paper discusses the "*Satisfactory Graph Partitioning Problem*" [Gerber and Kobler, European Journal of Operational Research 125 (2000) 283-291]. For any x ∈ X ⊆ V(G), x is "*satisfied*" when X contains at least half of x's neighbors. The set X is "*cohesive*" if all it's vertices are satisfied, and a graph is said to be "*satisfiable*" if there is a vertex partition into two or more non-empty cohesive sets. Such a partition is referred to as "*satisfactory partition*." Not all graphs have such a partitioning, for example complete graphs. In this paper, we present some necessary and sufficient conditions for satisfiable graphs. It is shown that no forbidden subgraph characterization exists for this class of graphs. Some special cases, for example, regular graphs and line graphs, are also discussed.

Keywords: Graph–satisfiability, vertex–partitions

1. Introduction

This paper deals with a graph-partitioning problem referred to as "**Satisfactory Graph Partitioning Problem (SGP)**" first introduced by Gerber and Kobler [2]. Consider a graph $G = (V, E)$ without loops or multiple edges, and a subset A of V. A vertex v in set A is said to be ***satisfied*** with respect to A if it has at least as many neighbors in A as in $V - A$, i.e., $|N(v) \cap A| \geq |N(v) - A|$, where $N(v)$ is the set of vertices adjacent to v. The set A is ***cohesive*** if every vertex in A is satisfied with respect to A, and a graph is said to be ***satisfiable*** if there is a vertex partition into two or more nonempty sets so that every vertex is satisfied with respect to the set in which it occurs. Such a partition is referred to as ***satisfactory partition***.

Not all the graphs have a satisfactory partitioning of vertices, as opposed to a similar problem known as "Unfriendly Graph Partitioning Problem (UGP)" [1], where a partition is said to be unfriendly if each vertex has as many or more neighbors outside the set in which it occurs than inside it. For example, complete graphs and complete bipartite graphs $\mathbf{K}_{p,q}$ (when p or q is odd) are not satisfiable. However if both p and q are even, $\mathbf{K}_{p,q}$ is satisfiable [2], and if a graph is separable or has a bridge that is not a pendant edge then the graph is satisfiable. Hence all trees with diameter greater than 2 are satisfiable. It is proved in [3] that every graph (that is not $K_{1,n}$) of girth at least 5 is satisfiable.

The complexity of SGP is still open, whereas there exists a polynomial time algorithm for finding an unfriendly partition for graphs.

In this paper, we have tried to further categorize satisfiable and unsatisfiable graphs. Section 2 discusses the relationship between satisfiability and connectivity of graphs. Section 3 presents results regarding categorization of satisfiable graph by their subgraphs. Section 4 treats special cases, for example, regular graphs and line graphs.

In the remainder of this paper, we will assume the following notation. Given a graph $G=(V,E)$, $n=|V|$ is the number of vertices, $m=|E|$ is the number of edges, $\delta(G)$ is the minimum degree and $\Delta(G)$ denotes the maximum degree of graph G. If $v \in V$ then $N(v)$ denotes the set of vertices adjacent to v, $v \notin N(v)$, and $\deg(v)=|N(v)|$. If $V' \subseteq V$ then $N(V')=\bigcup_{v \in V'} N(v)$, $G[V']$ is the graph induced by vertices in V' and $\deg_{V'}(v)=\deg_{G[V']}(v)=|N(v) \cap V'|$. Other notation will be introduced as needed. Since, disconnected graphs are trivially satisfiable, we will only consider connected graphs.

2. Satisfiability and Connectivity

In this section, we discuss the relation between the connectivity and satisfiability of a graph. An edge *cutset* of a connected graph G is a set $S \subseteq E(G)$ such that $G-S$ is disconnected. If no proper subset of S is a cutset, then S is called *minimal cutset.* If S has the minimum number of edges among all cutsets then S is called *minimum cutset* of G. Let V_1 and V_2 partition V. The edges of the cutset S which have one end vertex in V_1 and the other in V_2 is denoted as $S=\langle V_1,V_2\rangle$. The same notation will be used for the vertex partition formed by V_1 and V_2. The meaning of notation will be obvious by the context within which it is used.

A *Critical Cutset* $S=\langle V_1,V_2\rangle$ of a connected graph G is a minimal cutset, such that $|V_i|>1, i \in \{1,2\}$ and moving any vertex from one set to the other does not decrease the size of the resulting cutset.

Theorem 1 G is satisfiable if and only if it has a critical cutset.

Proof. Suppose G has a critical cutset $S=\langle V_1,V_2\rangle$ and there exists a vertex v which is not satisfied. Assume without loss of generality that $v \in V_1$. Then $\deg_{V_1}(v)<\deg_{V_2}(v)$ and we may form a new partition $S'=\langle V_1-\{v\},V_2 \cup \{v\}\rangle$ where $|V_1-\{v\}| \geq 1$. Now, $|S'|=|S|-\deg_{V_2}(v)+\deg_{V_1}(v)$. Since $\deg_{V_1}(v)<\deg_{V_2}(v)$, we must have that $|S'|<|S|$, contradicting the assumption that S is a critical cutset of G.

For the converse, consider a satisfiable graph G such that the cutset $S = \langle V_1, V_2 \rangle$ forms a satisfactory partition. Suppose that S is not a critical cutset, that is, there exists a vertex v, such that moving v from one set of the partition to another would decrease the size of cutset. Assume without loss of generality that $v \in V_1$. Then $S' = \langle V_1 - \{v\}, V_2 \cup \{v\} \rangle$ and $|S'| < |S|$. But $|S'| = |S| - \deg_{V_2}(v) + \deg_{V_1}(v)$ which means that $\deg_{V_1}(v) < \deg_{V_2}(v)$ and contradicts the assumption that $S = \langle V_1, V_2 \rangle$ is a satisfactory partition. ■

Edge connectivity $\kappa_1(G)$ of a graph G is the minimum number of edges whose removal from G results in a disconnected graph. The following result, also proven in [2], is a direct consequence of Theorem 1.

Corollary 1.1 A connected graph G is satisfiable if $\kappa_1(G) < \delta(G)$.

Vertex connectivity $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected graph.

Theorem 2 A graph G is satisfiable if $\kappa(G) \le \frac{\delta(G)}{2}$.

Proof. Suppose for a graph G, that $\kappa(G) \le \frac{\delta(G)}{2}$ and G is unsatisfiable. From Corollary 1.1, we may assume G is connected and that V' is a set of disconnecting vertices of G such that $1 \le |V'| \le \frac{\delta(G)}{2}$. Let A be the set of vertices of one of the components of $G - V'$ and let $B = V - V' - A$. The edge cutset $S = \langle B, A \cup V' \rangle$ partitions V into two subsets. Since $\forall v \in B, N(v) \cap A = \varnothing$, $N(v) - B \subseteq V'$, and thus $\deg_{V-B}(v) \le \frac{\delta(G)}{2} \le \deg_B(v)$. Hence, every vertex of B is satisfied. The only vertices in G, which may not be satisfied with respect to partition $\langle B, A \cup V' \rangle$, are those in V'. Now perform the following procedure on the partition.

While $\exists v \in V'$ such that $\deg_{V-B}(v) < \deg_B(v)$

Set $B \leftarrow B \cup \{v\}$, $V' \leftarrow V' - \{v\}$

This procedure will certainly terminate, as there is only a finite number of elements in V' and vertices are only moved from set V' to set B. Since every vertex of B was initially satisfied, no vertices are removed from B. Therefore every vertex of B is still satisfied. Also, all vertices of V' are now satisfied. Since at most $\frac{\delta(G)}{2}$ vertices were moved from set V' to set B, vertices of A are each adjacent to at most $\frac{\delta(G)}{2}$ vertices in B, and are satisfied. Thus, G is satisfiable. ■

3. Subgraph Characterizations

In this section we show that there is no forbidden subgraph categorization of satisfiable graphs. We also show the same holds for unsatisfiable graphs. First recall that a set of vertices is cohesive if all the vertices in that set are satisfied with respect to it. Formally, a set $X \subseteq V(G)$ is *cohesive* if $\deg_X(x) \geq \deg_{V-X}(x), \forall x \in X$. In an independent study [4], Hedetniemi et. al. have used the term *strong defensive alliance* for cohesive sets. A cohesive set is *strong cohesive* if the inequality 'greater than or equal to' is changed to 'strictly greater' and a subgraph is called cohesive if it is induced by a cohesive set. Every graph has a cohesive set $-V(G)$, itself. For every vertex $x \in V(G)$, $V(G) - x$ is a cohesive set if and only if x is not adjacent to a pendant vertex. If A and B are two cohesive sets then $A \cup B$ is also a cohesive set. A cohesive set X is *minimal* if no proper subset of X is cohesive. Every cohesive set contains a minimal cohesive subset.

Since $V(G)$ is itself cohesive, we define a cohesive set X to be *locally maximal* if $\forall v \notin X, X \cup \{v\}$ is not cohesive. If X is a locally maximal cohesive set of graph G then $V(G) - X$ is strong cohesive. Hence, a graph G is satisfiable if it has a locally maximal cohesive set of size less than n. The converse is not always true, for example, $\mathbf{C}_n, \forall n > 3$ is satisfiable but has no locally maximal cohesive set of size less than n.

Similarly, a *locally minimal* cohesive set is a cohesive set X, such that $\forall v \in X, X - \{v\}$ is not cohesive. Every minimal cohesive set is also locally minimal cohesive but a locally minimal cohesive set need not be minimal cohesive. A *minimum cohesive set* is a minimal set of smallest order. If X is a minimum cohesive set of a graph G, then $|X| \leq n - \frac{\delta(G)}{2}$ since, $\forall v \in X$, $\deg_X(v) \geq \frac{\delta(G)}{2} \geq \deg_{V-X}(v)$. If a graph G is satisfiable then by definition, it has at least two disjoint minimal cohesive sets (the converse of this is also true and is Lemma 1). Hence, if every minimal cohesive set of a graph G has at least

$\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices then G is unsatisfiable. We prove next that a minimum cohesive set of graph G has at most $\left\lfloor \frac{n}{2} \right\rfloor + 1$ vertices.

Proposition 1 For any graph G, of order n, if A is a minimum cohesive set then $|A| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1$.

Proof. Let A be a minimum cohesive set of a graph G and $B = V(G) - A$. Assume to the contrary that $|A| > \left\lfloor \frac{n}{2} \right\rfloor + 1$. If $\exists T \subseteq B$ and $v \in A$, such that T or $T \cup \{v\}$ is cohesive then $|T| + 1 \leq \left\lceil \frac{n}{2} \right\rceil - 1 < |A|$, a contradiction. Thus, there is a partition $\langle V_1, V_2 \rangle$ of $V(G)$ such that $\forall P \subseteq V_1$, P is not cohesive. Similarly, $\forall Q \subseteq V_2$, Q is not cohesive. Consider such a partition with the property that the size of edge-cutset separating V_1 and V_2 is minimum among all such partitions. Let S be the edge-cutset separating V_1 and V_2. Assume without loss of generality that $|V_1| \geq \left\lceil \frac{n}{2} \right\rceil$. Since V_1 is not cohesive, $\exists v \in V_1$ such that $\deg_{V_1}(v) < \deg_{V_2}(v)$. Consider the partition $\langle V_1 - \{v\}, V_2 \cup \{v\} \rangle$. Let S' be the edge-cutset separating $V_1 - \{v\}$ and $V_2 \cup \{v\}$ such that $|S'| = |S| - \deg_{V_2}(v) + \deg_{V_1}(v) < |S|$. Hence, at least one of the sets, $V_1 - \{v\}$ or $V_2 \cup \{v\}$, must be cohesive or contain a subset that is cohesive. Since $V_1 - \{v\}$ is not cohesive, $V_2 \cup \{v\}$ must be cohesive or contain a cohesive set, but then $|V_2 \cup \{v\}| \leq \left\lfloor \frac{n}{2} \right\rfloor + 1 < |A|$, a contradiction. ■

The following lemma is direct consequence of a result proved in [6].

Lemma 1 A graph G is satisfiable if and only if it has two disjoint cohesive sets.

To show the nonexistence of a forbidden subgraph characterization for satisfiable graphs, we first prove that there is no such characterization for cohesive subgraphs.

Lemma 2 There is no forbidden subgraph characterization for subgraphs induced by cohesive sets.

Proof. Suppose to the contrary that $G=(V,E)$ is a forbidden subgraph for graphs induced by cohesive sets. Let $|V|=n$ and $|E|=m$, and construct a graph $G'=(V',E')$ as follows:

$V'=V\cup X$, where $X=\{x_1,x_2,\ldots,x_m,y_1,y_2,\ldots,y_m\}$ is a set of independent vertices; and $E'=E\cup Y$, where $Y=\{a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_m\}$ such that if $e_i\in E$ is an edge between vertices s and t, then a_i is an edge between vertices s and x_i and b_i is an edge between vertices t and y_i. Hence, by construction, $\forall v\in V, \deg_V(v)=\deg_{V'-V}(v)$. Therefore, V is a cohesive set of graph G', contradicting our initial assumption. ■

Theorem 3 There is no forbidden subgraph characterization of satisfiable graphs.

Proof. Suppose to the contrary that $G=(V,E)$ is a forbidden subgraph for graphs induced by satisfiable graphs. Hence, G cannot be induced by any subset of a satisfiable graph. Construct a graph $G'=(V',E')$ as in the proof of Lemma 2, such that V is a cohesive set of G'. Select two vertices x_1 and x_2, arbitrarily, from the set of independent vertices $V'-V$, and add an edge between them. Since, by construction, both x_1 and x_2 were initially pendant vertices, therefore after addition of the edge, the set $\{x_1,x_2\}$ becomes a cohesive set in graph G'. Since G' has two disjoint cohesive sets, V and $\{x_1,x_2\}$, Lemma 1 implies G' is satisfiable, hence a contradiction. ■

The join of simple graphs G_1 and G_2, written $G_1\vee G_2$, is the graph obtained by adding the edges $\{xy : x\in V(G_1), y\in V(G_2)\}$.

Theorem 4 There is no forbidden subgraph characterization of unsatisfiable graphs.

Proof. Suppose to the contrary that $G=(V,E)$ is a forbidden subgraph for unsatisfiable graphs, that is G cannot be an induced subgraph of any unsatisfiable graph.

We construct a graph $G'=G\vee K_{n+1}$ where n is the number of vertices in G. Therefore, G' must be satisfiable, since G is an induced subgraph of G'. Let $\langle A,B\rangle$ be a satisfactory partition of G' and consider $v\in V(K_{n+1})$. Assume, without loss of generality, that $v\in A$. Since $\deg(v)=2n$, $|A|\geq n+1$. Then $|B|\leq n$ and no vertex of $V(K_{n+1})$ can be satisfied in B. Hence, $V(K_{n+1})\subseteq A$. Since $V(K_{n+1})\subseteq N(u), \forall u\in V(G)$, if $u\in B$, $\deg_A(u)\geq n+1>\deg_B(u)$.

Therefore, B must be an empty set, contradicting the assumption that $\langle A,B \rangle$ forms a satisfactory partition of G'. Hence, G' is unsatisfiable. ■

4. Special Cases

In this section, we deal with some special type of graphs. We prove that (3,4)-regular graphs, with a few exceptions, are satisfiable. A *(3,4)-regular graph* is a graph G with $3 \le \delta(G) \le \Delta(G) \le 4$. We also present conditions for the satisfiability of Eulerian graphs and line graphs.

Lemma 3 A set of vertices of a *(3,4)-regular graph* G is minimal cohesive if and only if it induces a cycle of G.

Proof. Suppose there exists a minimal cohesive set A in a (3,4)-regular graph G and let G' be the subgraph induced by A. First assume that G' is acyclic. Since A is minimal, G' must be connected and hence is a tree. Consider any leaf v of this tree, since the degree of v in G is either 3 or 4, v must be connected to at least 2 vertices outside A. Hence, A is not cohesive, contradicting our initial assumption. Therefore G' must contain a cycle.

Now assume that G' has more than one cycle. But then each cycle is also cohesive, contradicting that A is minimal. Hence, G' is a cycle.

For the converse, let A be the set of vertices of any induced cycle of a (3,4)-regular graph. Clearly, A is cohesive. Assume, to the contrary, that A is not minimal. Then a proper subset of A must be cohesive. But every proper subset of A induces a forest and has at least two vertices of degree less than 2 that are not satisfied, hence a contradiction. ■

Corollary A (3,4)-regular graph is satisfiable if and only if it has at least two vertex disjoint cycles.

Let Q be the set of graphs that have $n-3$ independent degree 3 vertices, where n is the number of vertices. A wheel W_n is a cycle on $n-1$ vertices plus a single vertex adjacent to all vertices of the cycle.

Lemma 4 If $\delta(G) \ge 3$, then G has two disjoint cycles if and only if $n \ge 6$, G is not a wheel, and G is not in Q.

Proof. If G has less than 6 vertices, then it cannot have vertex disjoint cycles. Since every cycle in a wheel contains a common vertex or has $n-1$ vertices, it cannot have vertex disjoint cycles. Suppose that $G \in Q$ and let A be the set of $n-3$ independent degree 3 vertices. Then every cycle in G must contain at least 2 vertices from $V-A$, hence G does not have vertex disjoint cycles.

We prove the converse by induction on the number of vertices. By case analysis, it can be seen that there are at least two vertex disjoint cycles in every graph G with $\delta(G) \geq 3$ and $n = 6$ when G is not a wheel and is not in Q. Assume the statement is true for all graphs with order $n \leq k$ for arbitrary $k \geq 6$.

Consider a graph G with $\delta(G) \geq 3$ and $n = k + 1$, G is not a wheel, and is not in Q. We pick a vertex v in G such that i) $\deg(v) = \delta(G)$, ii) Among all vertices of minimum degree, v maximizes the number of edges induced by $N(v) \cup \{v\}$. Consider the graph $G - v$. If $G - v$ is not a wheel and is not in Q, and $\delta(G - v) \geq 3$ then by induction hypothesis $G - v$ has at least two vertex disjoint cycles, hence G has at least two vertex disjoint cycles.

Assume that $\delta(G - v) < 3$, then $\deg_G(v) = 3$. Let $N(v) = \{v_1, v_2, v_3\}$, where the degree of at least one of v_i is 2 in graph $G - v$. Assume without loss of generality that $\deg_{G-v}(v_1) = 2$. Let G_1 be the graph obtained by adding edges in $G - v$ between the vertices of $N(v)$, such that $\delta(G_1) \geq 3$. Let $E' = E(G_1) - E(G)$, where $1 \leq |E'| \leq 2$.

Suppose that we cannot construct G_1 by adding edges because the vertices were already adjacent, then we have a triangle say vv_1v_2v in graph G such that v and v_1 are adjacent to exactly one vertex in $V' = V - \{v, v_1, v_2\}$. If there is any cycle in V', then the graph G has 2 vertex disjoint cycles. Now assume that the graph $G[V']$ is acyclic. Since $\delta(G) = 3$, every vertex in $G[V']$ with degree less than 2 must be connected to at least 2 vertices of the triangle, hence, $G[V']$ must be a path with each internal vertex having an edge to v_2. Therefore G is a wheel, contradicting our assumption.

Let $G_1 \in Q$ and let V_1 be the set of $k - 3$ independent vertices and $V_2 = V - V_1$. If $V_1 \cap N(v) = \varnothing$ then $G \in Q$, whereas if $V_2 \cap N(v) = \varnothing$ then $|E'| = 0$, a contradiction. Since $\forall w \in V_2, \deg_{V_1}(w) \geq 3$, therefore $\forall e \in E', e = xy \Rightarrow x \in V_1 \wedge y \in V_2$. Also, since $\forall w \in V_1, \deg(w) = 3$, every vertex in V_1 is end vertex of at most one edge in E'. Now consider two cases. Case 1. $|E'| = 1$: Let $e = v_1v_2 \in E'$ such that $v_1 \in V_1$ and $v_2 \in V_2$, then there are two vertex disjoint cycles in G, a triangle T (where $T = vv_1v_3v$, if $v_3 \in V_2$ and $T = vv_2v_3v$, if $v_3 \in V_1$) and $wxyzw$ where $w, y \in V_1 - T$ and $x, z \in V_2 - T$. Case 2. $|E'| = 2$: Let $e_1, e_2 \in E'$ where $e_1 = v_1v_2$ and $e_2 = v_3v_2$. Then the only possibility is that $|V_1| \geq 4$, $v_1, v_3 \in V_1$ and $v_2 \in V_2$. Again there are two vertex disjoint cycles in G, vv_1pv_3v (where $p \in V_2 - \{v_2\}$) and $wxyzw$ where $w, y \in V_1 - N(v)$ and $x, z \in V_2 - \{p\}$.

Let G_1 be a wheel such that $X = \{x_1, x_2, \ldots, x_{k-1}\}$ forms a cycle C, and y is a vertex adjacent to every vertex of X. Since $\forall x_i \in X - N(v)$, $\deg(x_i) = 3$ and $N(x_i) \cup \{x_i\}$ induces 5 edges in G, by choice of v, $N(v) \cup \{v\}$ must induce at least 5 edges in G. But this is possible only if $\exists x_j, x_{j+1} \in N(v)$. If $x_j x_{j+1} \in E(G)$ then there are two vertex disjoint cycles $v x_j x_{j+1} v$ and $y x_{j+2} x_{j+3} y$ in G_1. Otherwise $vy \in E(G)$ and hence G is a wheel, a contradiction.

We may now assume that G_1 is not a wheel and is not in Q. Hence by induction hypothesis, G_1 has two vertex disjoint cycles. If these cycles do not include the edges in E', then G has two vertex disjoint cycles. If any of these cycles in G_1 include a path (assume $v_1 v_2$) consisting of edges in E', then it can be extended in G by replacing the path $v_1 v_2$ by edges $v_1 v$ and $v v_2$. Hence, G has two vertex disjoint cycles.

Now assume that $G - v$ is a wheel such that $X = \{x_1, x_2, \ldots, x_{k-1}\}$ forms a cycle C, and y is a vertex adjacent to every vertex of X. Then, in G, v must be adjacent to at least two vertices $x_i, x_j \in X$. Let x_m, x_{m+1} be two adjacent vertices in one of the $x_i - x_j$ paths in C, then $y x_m x_{m+1} y$ and $v x_i - x_j v$ forms two vertex disjoint cycles in G.

Finally, assume that $G - v \in Q$. Let $A = \{a_1, a_2, \ldots, a_{k-3}\}$ be the set of $k-3$ independent degree 3 vertices and $B = \{b_1, b_2, b_3\}$ be the remaining 3 vertices. Then v must be adjacent to at least one vertex $a_i \in A$, otherwise $G \in Q$. If v is connected to any vertex in B, say b_1, then G has at least two vertex disjoint cycles, $v a_i b_1 v$ and the other formed by b_2, b_3 and any two vertices in A other than a_i. If v is not connected to any vertex in B, let $a_i, a_j \in N(v)$, then again there are two vertex disjoint cycles, $v a_i b_1 a_j v$ and $b_2 a_p b_3 a_q b_2$ where a_p and a_q are vertices in A other than a_i and a_j. The vertices a_p and a_q always exist for all $k > 6$. When $k = 6$ then either there are vertex disjoint cycles, $v a_i b_1 a_j v$ and $a_p b_2 b_3 a_p$ or $G \in Q$, contradicting the hypothesis. ■

Theorem 5 A (3,4)-regular graph G is satisfiable if and only if $n \geq 6$, G is not a wheel and G is not in Q.

Corollary 5.1 Every (3,4)-regular graph of order $n \geq 8$ is satisfiable.

Corollary 5.2 Every 4-regular graph except $\mathbf{K}_5$ is satisfiable.

Corollary 5.3 Every 3-regular graph except $\mathbf{K}_{3,3}$ and $\mathbf{K}_4$ is satisfiable.

We believe, but have been unable to prove the following generalization of Corollary 5.2 is true.

Conjecture Every finite $2k$-regular graph with more than $2k+1$ vertices is satisfiable.

We prove a weaker result that all triangle free $2k$-regular graphs are satisfiable. The proof follows a similar reasoning as in [5]. We define a set A to be *degenerate* if $\forall S \subseteq A, \exists v \in S$ such that $\deg_S(v) \le \frac{\deg(v)}{2}$. It is *strong degenerate* if the inequality is strong. If a set A is (strong) degenerate, then $\forall S \subseteq A,$ S is also (strong) degenerate. If A is not strong degenerate then $\exists S \subseteq A,$ such that $\forall v \in S, \deg_S(v) \ge \frac{\deg(v)}{2}$, i.e. A contains a cohesive set. Similarly, if A is not degenerate then A contains a strong cohesive set.

Theorem 6 Every triangle free Eulerian graph is satisfiable.
Proof. Let G be a triangle-free Eulerian graph. Assume to the contrary that G is unsatisfiable. Consider a partition $\langle A,B\rangle$ of $V(G)$ such that A is degenerate containing a cohesive set, say T. Since every minimal cohesive set is degenerate, such a partition always exists. Let the partition $\langle A,B\rangle$ be such that the edge cutset $S=\langle A,B\rangle$ is minimum among all such partitions. Further assume that A is minimal subject to these properties. Since A contains a cohesive set, $|A| \ge 2$. Since A is degenerate, there is a vertex $v \in A$ such that $\deg_A(v) \le \frac{\deg(v)}{2}$, hence $|B| \ge \frac{\delta(G)}{2} \ge 1$. Suppose $|B| = 1$, and let $q \in B$, then $\exists r \in A$ such that $\deg(r) = 2$ and $qr \in E(G)$. Consider the partition $\langle A-\{r\}, B \cup \{r\}\rangle$. By definition $A-\{r\}$ is degenerate and the size of the new edge cutset is equal to $|S|$. By minimality of A, the only alternative is that $A-\{r\}$ does not contain any cohesive set, that is $\exists s \in A-\{r\}$ such that $\deg_{A-\{r\}}(s) < \frac{\deg(s)}{2}$. Since G is Eulerian, this is possible only if $\deg_{A-\{r\}}(s) = 0$ and $\deg(s) = 2$, but this implies that there is a triangle qrs in G, a contradiction. Hence $|B| \ge 2$. Recall that if B is not strong degenerate then it

contains a cohesive set, say C. But then there are two vertex-disjoint cohesive sets C and T in G, a contradiction. So we may assume that B is strong degenerate, i.e. $\exists x \in B$ such that $\deg_B(x) < \frac{\deg(x)}{2}$. Let $D = \left\{ v \in A \mid \deg_A(v) \leq \frac{\deg(v)}{2} \right\}$. Since A is degenerate, $D \neq \emptyset$. We claim that for any cohesive set $T' \subseteq A$, $D \subseteq T'$. Suppose not. Then there exists a vertex $v \in A - T'$ such that $\deg_A(v) \leq \frac{\deg(v)}{2}$. Hence the size of cutset $S' = \langle A - \{v\}, B \cup \{v\} \rangle$ is at most $|S|$. By definition $A - \{v\}$ is degenerate and since $T' \subseteq A - \{v\}$, $A - \{v\}$ contains a cohesive set, which is contradiction since A is a minimal such set. Hence $D \subseteq T'$, as claimed. This also implies that $\forall v \in D, \deg_A(v) = \frac{\deg(v)}{2}$. Hence A is a cohesive set.

Now we claim that $D \subseteq N(x)$, $\forall x \in B$ such that $\deg_B(x) < \frac{\deg(x)}{2}$. Suppose not. Consider the partition $\langle A \cup \{x\}, B - \{x\} \rangle$, the cutset $S' = \langle A \cup \{x\}, B - \{x\} \rangle$ is strictly smaller than $|S|$. Hence $A \cup \{x\}$ can not be degenerate, i.e., there exists a cohesive set $T' \subseteq A$ such that $T' \cup \{x\}$ is a strong cohesive set and $\forall v \in T' \cup \{x\}, \deg_{T' \cup \{x\}}(v) > \frac{\deg(v)}{2}$. Since $D \subseteq T'$, and $\forall v \in D, \deg_A(v) = \frac{\deg(v)}{2}$, $D \subseteq N(x)$.

Let $y \in D$ and consider the partition $\langle A - \{y\}, B \cup \{y\} \rangle$. The size of cutset $S'' = \langle A - \{y\}, B \cup \{y\} \rangle$ is equal to $|S|$. By minimality of A, the only alternative is that $A - \{y\}$ does not contain any cohesive set and hence, $\exists z \in A - \{y\}$ such that $\deg_{A-\{y\}}(z) < \frac{\deg(z)}{2}$, this implies that $z \in D$ and $yz \in E(G)$. But since $D \subseteq N(x)$, xyz is a triangle in G contradicting the assumption that G is triangle free. ■

Corollary 6.1 Every triangle free $2k$-regular graph is satisfiable.

Theorem 7 If a graph G has two non-adjacent vertices of maximum degree then the Line graph $L(G)$ of G is satisfiable.

Proof. Let $u,v \in V(G)$, such that $uv \notin E(G)$ and $\deg(u)=\deg(v)=\Delta(G)$. Let maximum degree of $L(G)$ be Δ', then, $\Delta \geq \frac{\Delta'}{2}+1$. Since the edges incident to vertex u in G form a clique $\mathbf{K}_\Delta$ in $L(G)$, the set of vertices in this clique is a cohesive set in $L(G)$. Similarly, the set of vertices corresponding to the edges incident to v is also a cohesive set in $L(G)$. Since both cohesive sets are disjoint, by Lemma 1 $L(G)$ is satisfiable. ■

Theorem 8 If a graph G has a maximum-degree vertex not adjacent to any degree 2 vertex then its line graph $L(G)$ is satisfiable if and only if G is not a star, $\mathbf{K}_{1,n-1}$.

Proof. Let $v \in V(G)$, such that $\deg(v)=\Delta(G)$ and $\forall w \in N(v), \deg(w) \neq 2$. Then, by the proof of Theorem 7, the set of vertices V_1 corresponding to the edges incident to v is a cohesive set in $L(G)$. The set $V_2 = V(L(G)) - V_1 = \varnothing$ if and only if G is a star. Since complete graphs are not satisfiable, $L(G)$ is not satisfiable if G is a star. If $V_2 \neq \varnothing$ then every vertex in V_2 is adjacent to at most 2 vertices of V_1. Also, since $N(v)$ has no degree 2 vertex, $\forall u \in V_2, |N(u) \cap V_2| \geq 2 \geq |N(u) \cap V_1|$. Hence V_2 is also a cohesive set in $L(G)$. By Lemma 1, $L(G)$ is satisfiable. ■

5. References

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