PARTITION INTO GLOBAL DEFENSIVE ALLIANCES(PGDA) is NP-Complete.

NOT ALL EQUAL 3SAT (NAE3SAT)
Input: A set $U = \{u_1, u_2, ..., u_n\}$ of variables and a collection $C = \{C_1, C_2, ..., C_m\}$ of clauses over U, where each clause contains exactly three literals (variables or their complements}, with no literal appearing more than once in any given clause.

Question: Is there a truth assignment that makes one or two (but not all three) literals true in each clause?

We may assume that each literal appears in at least one of the clauses, otherwise, for each literal u_j that does not appear in any of the clauses, we can add another variable y and two clauses $C_1' = \{u_j, \bar{u}_j, y\}$ and $C_2' = \{u_j, \bar{u}_j, \bar{y}\}$. These two clauses are satisfied by any truth assignment and do not affect the truth assignment of the original problem.

Theorem: Given a graph G, the problem of deciding whether the graph G has a partition into global defensive alliances is NP Complete.

Proof: Given an instance of NAE3SAT with n variables and m clauses, we transform it into an instance of PGDA by constructing a graph $G = (V, E)$ as follows:

For a literal $u \in U \cup \bar{U}$, let $C(u)$ be the set of clauses that contains u. Let $V = Q \cup X \cup R \cup T$, where $Q = \{q(u), u \in (U \cup \bar{U})\}$, $X = \{x_i, 1 \le i \le n\}$, $R = (\cup_{u \in (U \cup \bar{U})} R(u))$, and $T = \cup_{1 \le j \le m} T_j$. For all $u \in U$, $R(u) = \{r_i(u), 1 \le i \le |C(u)| + 2\}$, and for all $\bar{u} \in \bar{U}$, $R(u) = \{r_i(u), 1 \le i \le |C(u)|\}$. Also, for all $C_j \in C$, $T_j = \{t_j(a), t_j(b), t_j(c) \mid C_j = \{a, b, c\}\}$. For each literal $u \in U \cup \bar{U}$, we create a star $S(u)$, where $V(S(u)) = \{q(u)\} \cup R(u)$ and the vertex $q(u)$ forms the center of the star. For each $x_i \in X$, we add edges $x_i q(u_i)$ and $x_i q(\bar{u}_i)$ in graph G. For each clause $C_j \in C$, we setup a triangle T_j in V and for each vertex $t_j(u) \in T_j$, add an edge $q(u)t_j(u)$ in graph G.
The order of the constructed graph, $|V| = 5n + 6m$ and the size of the graph, $|E| = 4n + 9m$, which is polynomially related to the size of NAE3SAT problem.

We now claim that the constructed graph G has a partition into global defensive alliances if and only if the given instance of NAE3SAT has a satisfying truth assignment. The proof of the claim is as follows:

$\Rightarrow$ Suppose that the given instance of NAE3SAT has a satisfying truth assignment $f : U \to \{0,1\}$. We define a partition of the vertex set $V = A_1 \cup A_2$ as follows:
$A_1 = \bigcup_{u_i \in (U \cup \bar{U}) \wedge f(u_i)=1} \{q(u_i)\} \cup \{x_i \mid u_i \in U\} \cup R(\bar{u}_i) \cup \left(\bigcup_{1 \le j \le m} (\{t_j(u_i)\} \cap T_j)\right)$ and

$A_2 = V - A_1$. We now show that both A_1 and A_2 are global defensive alliances in graph G. Consider any $v \in A_r, r \in \{1,2\}$ and consider five cases:

Case 1: $v = q(u_i)$ for some $u_i \in U$. From the partitioning scheme, $N(v) \cap A_r = \{x_i\} \cup \left(\bigcup_{1 \le j \le m} (\{t_j(u_i)\} \cap T_j)\right)$, i.e., $|N(v) \cap A_r| = 1 + |C(u_i)|$. Similarly, $N(v) \cap (V - A_r) = R(u_i)$, i.e., $|N(v) \cap (V - A_r)| = 2 + |C(u_i)|$. Hence $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$.

Case 2: $v = q(u_i)$ for some $u_i \in \overline{U}$. Again, from the partitioning scheme, $N(v) \cap A_r = \bigcup_{1 \le j \le m} (\{t_j(u_i)\} \cap T_j)$, i.e., $|N(v) \cap A_r| = |C(u_i)|$. Similarly, $N(v) \cap (V - A_r) = R(u_i)$, i.e., $|N(v) \cap (V - A_r)| = |C(u_i)|$. Hence, $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$.

Case 3: $v \in R(u_i)$ for some $u_i \in U \cup \overline{U}$. Then by construction and partitioning scheme, $N(v) \cap A_r = \varnothing$ and $N(v) \cap (V - A_r) = \{q(u_i)\}$. Hence, $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$.

Case 4: $v = x_i$. Then $N(v) \cap A_r = \{q(u_i)\}$ and $N(v) \cap (V - A_r) = \{q(\overline{u}_i)\}$. Hence, $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$.

Case 5: $v = t_j(u_i)$ for some $u_i \in U \cup \overline{U}$. By construction and partitioning scheme, $N(v) \cap A_r \supseteq \{q(u_i)\}$ and by the property of satisfying truth assignment of NAE3SAT, $N(v) \cap (V - A_r) \subseteq T_j - \{t_j(u_i)\}$. Hence, $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$.

Since for all $v \in A_r$ where $r \in \{1,2\}$, $|N(v) \cap A_r| \ge |N(v) \cap (V - A_r)| - 1$, A_r is a defensive alliance. It is also easy to see that $N[A_r] = V$. Hence, both A_1 and A_2 are global defensive alliances and they partition the vertex set of graph G.

$\Leftarrow$ Suppose now that the constructed graph G has a vertex partition A_1, A_2 such that both A_1 and A_2 are global defensive alliances. Since each vertex $x_i \in X$ is adjacent to only two vertices $q(u_i)$ and $q(\overline{u}_i)$, exactly one of these vertices must be in the same set A_r, $r \in \{0,1\}$, otherwise $N[V - A_r] \subseteq V - \{x_i\}$, which is contrary to $V - A_r$ being a global defensive alliance. Suppose that $q(u_i) \in A_r$ for some $u_i \in U \cup \overline{U}$ and $r \in \{0,1\}$. It is easy to see that $R(u_i) \cap A_r = \varnothing$. Suppose to the contrary and let $w \in R(u_i) \cap A_r$. Since $N[w] = \{q(u_i), w\} \subseteq A_r$, $N[V - A_r] \subseteq V - \{w\}$, which is contrary to $V - A_r$ being a global defensive alliance. This also implies that for all $q(u_i) \in A_r$, $N(q(u_i)) \cap (V - A_r) \supseteq R(u_i)$. Consider two cases:

Case 1: $u_i \in U$, then by construction, $|R(u_i)| = |C(u_i)| + 2$. Thus by above implication, we have, $|N(q(u_i)) \cap (V - A_r)| \ge |C(u_i)| + 2$. Since $|N[q(u_i)]| = 2|C(u_i)| + 4$, and since A_r is a

defensive alliance, we must have
$N[q(u_i)] \cap A_r = N[q(u_i)] - R(u_i) = \{q(u_i), x_i\} \cup \left(\bigcup_{1 \le j \le m} (\{t_j(u_i)\} \cap T_j)\right)$.
Case 2. $u_i \in \overline{U}$, then by construction, $|R(u_i)| = |C(u_i)|$. Also by Case 1,
$x_i \in N(q(u_i)) \cap (V - A_r)$. Thus $|N(q(u_i)) \cap (V - A_r)| \ge |C(u_i)| + 1$. Since
$|N[q(u_i)]| = 2|C(u_i)| + 2$ and since A_r is a defensive alliance, we must have
$N[q(u_i)] \cap A_r = N[q(u_i)] - R(u_i) - \{x_i\} = \{q(u_i)\} \cup \left(\bigcup_{1 \le j \le m} (\{t_j(u_i)\} \cap T_j)\right)$.

The above arguments show the following lemma.

Lemma 1: For all $u_i \in U \cup \overline{U}$, if $q(u_i) \in A_r$ for $r \in \{0,1\}$, then for all $t_j(u_i) \in T$, $1 \le j \le m$, $t_j(u_i) \in A_r$.

We now define a truth assignment, $f : U \to \{0,1\}$, such that $f(u_i) = 1$ if and only if $q(u_i) \in A_1$. We claim that *f* is a truth assignment that makes one or two (but not all three) literals true in each clause. Suppose to the contrary and let $C_j = \{a, b, c\}$ be a clause such that $f(a) = f(b) = f(c)$. From the definition of *f*, $\{q(a), q(b), q(c)\} \subseteq A_r$, $r \in \{0,1\}$ and from Lemma 1, $T_j = \{t_j(a), t_j(b), t_j(c)\} \subseteq A_r$. By construction of graph G,
$N[T_j] = N[t_j(a)] \cup N[t_j(b)] \cup N[t_j(c)] = T_j \cup \{q(a), q(b), q(c)\}$, which is contrary to $V - A_r$ being a global defensive alliance.