On X Free Covers

Khurram H. Shafique and Ronald D. Dutton
School of Computer Science
University of Central Florida
Orlando, FL USA 32816
khurram@cs.ucf.edu, dutton@cs.ucf.edu

Abstract

For an arbitrary graph $G = (V, E)$, let X be a graphical property that can be possessed, or satisfied by the subsets of V. For example, being a clique (maximal complete subgraph), a maximal independent set, an edge, a closed neighborhood, a minimal dominating set, etc. Let $C_X = \{A | A \subseteq V,\ |A| > 1$, and A possesses or satisfies property $X\}$. A set S is an X cover (or X free) if $A \cap S$ (or $A \cap (V - S)$) is not empty for every $A \in C_X$. Further, S is an X free cover when S is both X free and an X cover.

Many vertex-partitioning problems can be viewed as that of finding an X free cover. In this paper, we present properties of X free covers and investigate the conditions for their existence in some special cases. This provides an underlying unification theory for these and other similar properties.

Keywords: vertex cover, edge cover, independent set, vertex-partition.

1 Introduction

Vertex partitioning problems are widely studied in discrete optimization. Mainly, these problems involve splitting the vertex set into two or more sets such that each set satisfies a given criteria. Many of these problems are special cases of the classical **Set Splitting** problem. Given a finite set T and a collection C of subsets of T, the problem of finding a partition of T into two subsets T_1 and T_2, such that no subset in C is entirely contained in either T_1 or T_2 is called Set Splitting [5].

Let X denote an arbitrary property of a set of vertices in a graph $G = (V, E)$. Examples of X include: is a clique (maximal complete subgraph), is a maximal independent set, is an edge, is a closed neighborhood, is a minimal dominating set, is acyclic, is a tree, is planar, etc. Further, assume that $C_X = \{A | A \subseteq V$, $|A| > 1$, and A possesses or satisfies property $X\}$. Typical questions that arise are: What are the minimum and maximum sets in C_X for a given graph?, What are the upper and lower bounds on the cardinalities of these sets?, Can the vertex set of a graph be partitioned into two ore more sets in C_X?, etc.

In this paper, we consider a special case of Set Splitting problem, where the finite set T is the vertex set V of a simple graph $G = (V, E)$, and the collection C is defined by the subsets of V that satisfy some property X, i.e., C_X. The problem is then to partition the vertex set of a graph into two sets, such that, none of these sets contains a subset that satisfies property X. Such a partition is called an X *free partition.* A bipartition of the vertex set of a graph is an X *partition* if both of the sets in the partition satisfy property X.

A set $S \subseteq V$ is an X *free* set, if for all $A \in C_X$, $A - S \neq \emptyset$, i.e., S does not contain any set having property X as a subset. For example, if X is the property of being an edge then an X free set (or edge free set) is an independent set. Similarly, if X is the property of being independent then an X free set is a clique. An X free set S is maximal if $\forall v \notin S$, $\exists A \subseteq S$ such that $A \cup \{v\} \in C_X$. A maximum X free set is a maximal X free set of largest cardinality.

A set $S \subseteq V$ is an X *cover* if for all $A \in C_X$, $A \cap S \neq \emptyset$, i.e., S contains at least one vertex from each set of G that possesses property X. For example, if X is the property of being an edge then an X cover set (or edge cover set) S is a set that contains at least one end vertex of every edge of G. Note that such a set is called vertex cover set or hitting set in the literature. An X cover set S is minimal if no proper subset of S has property X. A minimum X cover set is a minimal X cover of smallest cardinality.

The following theorem shows the duality between X covers and X free sets.

Theorem 1. *$S \subseteq V$ is an X cover if and only if $V - S$ is X free.*

Proof. A set S is an X free set if and only if, for every set $A \in C_X$, $A - S \neq \emptyset$ if and only if, for every set $A \in C_X$, $A \cap (V - S) \neq \emptyset$ if and only if $V - S$ is an X cover. □

A set S is an X *free cover* if S is both X free and an X cover. Equivalently, S is an X free cover if for all $A \in C_X$, $A \cap S \neq \emptyset$ and $A \cap (V - S) \neq \emptyset$. Thus, we have the following:

Theorem 2. *A set S is an X free cover if and only if $V - S$ is an X free cover.*

Corollary 3. *A graph G has an X free partition if and only if G has an X free cover.*

In general, the decision version of Splitting Set problem is NP-Complete. The case where all subsets in the collection C are of the same size k is also NP-Complete, for all $k > 2$. The problem is equivalent to a variant of Satisfiability problem called Monotone NAESAT problem, where the clauses contain only unnegated literals and the goal is to decide whether a truth assignment exists such that all clauses have at least one true literal and one false literal.

The case, where the collection C is defined by some graph property X has also been investigated for a number of cases of X. The case where X is the property of having a path of length greater than some fixed k is studied in [3]. In [6], it is shown that the problem of determining whether an arbitrary graph has a 'cycle' free cover is NP Complete. Achlioptas [1] proved that if X is any fixed graph G of order greater than 2 then the problem of deciding whether an arbitrary graph has an X free cover, is NP Complete. An additive induced hereditary graph family is a set of graphs that is closed under taking vertex disjoint unions and taking induced subgraphs. In a related work, Farrugia [4] showed that vertex partitioning into fixed additive induced hereditary graph families is NP Hard.

This work provides a unified setting in which these and other similar concepts can be studied. We begin by exploring the existence and complexity of computing X free covers for some special cases of property X.

2 Basic Properties

Let X and Y be any two properties that can be satisfied or possessed by the subsets of V.

Proposition 4. *If $X \implies Y$, i.e., $C_X \subseteq C_Y$ then every Y free cover is also an X free cover.*

A property X is called *additive* if C_X is closed under unions, whereas it is called *hereditary* if C_X is closed under taking subsets. For example, the property 'is acyclic' is both an additive and a hereditary property. A property X is $k-$ary if it is only defined on the subsets of cardinality greater than or equal to k. For example, the property 'is a cycle' is a 3-ary property in a simple graph. A $k-$ary property is hereditary if C_X is closed under taking subsets of cardinality greater than or equal to k.

Theorem 5. *For an additive hereditary $k-$ary property X, G has an X free cover if and only if for all $A \in C_X$, $|A| < 2k-1$.*

Proof. Let X be an additive hereditary $k-$ary property and let $V = S \cup T$ be an X free partition in a graph G. Assume to the contrary that there exists a set $A \in C_X$ such that $|A| \geq 2k-1$. Without loss of generality, it may be assumed that $A \cap S \geq k$. Since X is hereditary, $A \cap S \in C_X$, which contradicts S being X free.

Assume now that for all $A \in C_X$, $|A| < 2k-1$ and that G does not have an X free cover. Let S be a minimal X cover. Since G does not have an X free cover, there must exist a set $A \subseteq S$ such that $A \in C_X$. By the minimality of set S, for every vertex $v \in S$, there is a set $B \in C_X$ such that $S \cap B = \{v\}$. Then, by additivity of X, $A \cup B \in C_X$. Since $|B \cap A| \leq 1$ and $|B| \geq k$ and $|A| \geq k$, we have that $|A \cup B| \geq 2k - 1$, a contradiction. □

Let X be a $k-$ary property. A property $\overline{X}$ is the *complement* of X, if for every set $A \subseteq V$ of cardinality greater than or equal to k, $A \in C_X \oplus C_{\overline{X}}$, where $P \oplus Q = (P \setminus Q) \cup (Q \setminus P)$.

Proposition 6. *If a $k-$ary property X is hereditary then $\overline{X}$ is additive.*

Proposition 7. *If X is hereditary then G has an $\overline{X}$ free cover if and only if one of the following is true:*

(i) $|V| < 2k - 1$,

(ii) *there exists a set $A \in C_X$, such that $|A| \geq |V| - k + 1$, or*

(iii) *the graph G has an X partition.*

Theorem 8. *If X is a hereditary $k-$ary property and $|V| > 4k - 3$ then G has an X free cover if and only if G does not have an $\overline{X}$ free cover.*

Proof. Let X be a hereditary $k-$ary property and let G be a graph with $|V| > 4k - 3$.
$\Longrightarrow$ Let $A \subseteq V$ be an X free cover in the graph G. From Theorem 2, it may be assumed that $|A| \geq 2k - 1$ and thus $A \in C_{\overline{X}}$. Assume to the contrary that G has an $\overline{X}$ free cover P. Once again, from Theorem 2, $V - P$ is also an $\overline{X}$ free cover. Since $A \in C_{\overline{X}}$, $A \cap P \neq \emptyset$ and $A \cap (V - P) \neq \emptyset$. Assume, without loss of generality, that $|A \cap P| \geq k$. Thus, $P \in C_X$. Since X is a hereditary property, $A \cap P \in C_X$, which contradicts A being an X free cover.
$\Longleftarrow$ Let $A \subseteq V$ be an $\overline{X}$ free cover in the graph G. Once again, it may be assumed that $|A| \geq 2k - 1$ and thus $A \in C_X$. Assume to the contrary that G has an X free cover S. Without loss of generality, it may be assumed that $S \cap A \geq k$. Since X is hereditary, $S \cap A \in C_X$, which contradicts S being X free. □

A property X is *superhereditary* if whenever a set S has property X, so does every proper superset $S' \supset S$. For example, the property 'contains a triangle' is superhereditary.

Proposition 9. *A property X is superhereditary if and only if its complement $\overline{X}$ is hereditary.*

Let Z be the property of being an X cover for some property X, i.e., $C_Z = \{A | A \subseteq V$ and A is an X cover$\}$. By the definition of X free cover, we have that G has an X free cover if and only if G has a Z partition.

Theorem 10. *If Z is the property of being an X cover for some superhereditary property X then G has a Z free cover if and only if G has an X partition.*

Proof. Let X be a superhereditary property and Z be the property of being an X cover. Consider an X partition $V = A \cup B$. By definition of X cover, every X cover $S \in C_Z$ must have nonempty intersections with both sets A and B, which implies that both A and B are Z free covers.

Now, assume that G has a Z free cover T. By Theorem 2, $U = V - T$ is also a Z free cover. We claim that $\{T, U\} \subseteq C_X$. Assume to the contrary that $T \notin C_X$, i.e., $T \in C_{\overline{X}}$. Since X is superhereditary, $\overline{X}$ is hereditary, and hence, T is X free. Thus, U is an X cover, i.e., $U \in C_Z$, which contradicts U being Z free. Hence $T \in C_X$ and similarly $U \in C_X$. □

3 Special Cases

3.1 When X='is an Edge'

If X is the property of being an edge then an *edge* free set is an independent set, and an *edge* cover is a set that contains at least one end vertex of every edge of the graph. The following well known result is immediate within this framework.

Proposition 11. *A graph has an edge free cover if and only if it is bipartite.*

3.2 When X='is Independent'

If X is the property of being independent then an *independent* free set is a clique.

Proposition 12. *A graph G has an 'independent set' free cover if and only if its complement $\overline{G}$ is bipartite.*

3.3 When X='is Neighborhood'

The *open neighborhood* of a vertex v is the set $N(v) = \{u | uv \in E\}$, whereas *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$.

If X is the property of being a closed neighborhood, then an X cover is a set S such that every vertex $v \in V$ is either an element of S or is adjacent to an element of S, i.e., S is a *dominating set.* By Theorem 1, an X free cover is a dominating set S, such that $V - S$ is also a dominating set. The following is a direct consequence of a theorem in [8].

Proposition 13. *If X is the property of being a closed neighborhood, then a graph G has an X free cover if and only if G does not have any isolated vertex.*

If X is the property of being an open neighborhood, then an X cover is a set S such that every vertex $v \in V$ is adjacent to an element of S, i.e., S is a *total dominating set* [7]. Once again, an X free cover is a total dominating set S, such that $V - S$ is also a total dominating set.

Define a problem ONFC(OPEN NEIGHBORHOOD FREE COVER) to be the problem of deciding whether a given graph has an 'open neighborhood' free cover. We show that ONFC is NP-Complete by giving a polynomial transformation from NAE3SAT problem, which is defined as follows:

INSTANCE NAE3SAT: a set $U = \{u_1, u_2, \ldots, u_n\}$ of variables and a collection $C = \{C_1, C_2, \ldots, C_m\}$ of clauses over U, where each clause contains exactly three literals (variables or their complements), with no literal appearing more than once in any given clause.

QUESTION: Is there a truth assignment that makes one or two (but not all three) literals true in each clause?

It may be assumed that each literal appears in at least one of the clauses, otherwise, for each literal u_j that does not appear in any of the clauses, one can add another variable y and two clauses $C'_1 = \{u_j, \overline{u_j}, y\}$ and $C'_2 = \{u_j, \overline{u_j}, \overline{y}\}$. These two clauses are satisfied by any truth assignment and do not affect the truth assignment of the original problem.

Theorem 14. *Given a graph G, the problem of deciding whether the graph G has an 'open neighborhood' free cover is NP-Complete.*

Proof. Given an instance of NAE3SAT with n variables and k clauses, we transform it into an instance of ONFC by constructing a graph $G = (V, E)$ as follows:

Let $V = R \cup S \cup \overline{S} \cup T$, where $R = \{r_i, 1 \leq i \leq m\}$, $S = \{s_i, 1 \leq i \leq n\}$, $\overline{S} = \{\overline{s_i}, 1 \leq i \leq n\}$, and $T = \{t_i, 1 \leq i \leq n\}$. For each variable $u_i \in U$, create a $P_3 = s_i t_i \overline{s_i}$. For each clause $C_i \in C$, create a vertex r_i in V such that $N(r_i) = \{s_j | u_j \in C_i\} \cup \{\overline{s_k} | \overline{u_k} \in C_i\}$.

The order of the constructed graph, $|V| = 3n + m$ and the size of the graph, $|E| = 3n + 3m$, which is polynomially related to the size of the NAE3SAT problem.

We claim that the constructed graph G has an 'open neighborhood' free cover if and only if the given instance of NAE3SAT has a satisfying truth assignment. The proof of the claim is as follows:

$\Longrightarrow$ Suppose that the given instance of NAE3SAT has a satisfying truth assignment $f : U \longrightarrow \{0, 1\}$. Define a partition of vertex set $V = A \cup B$ as follows: $A = R \cup \{s_i | f(u_i) = 1\} \cup \{\overline{s_i} | f(u_i) = 0\}$ and $B = V - A$. To show that $\forall v \in V$, $N(v) \cap A \neq \emptyset$ and $N(v) \cap B \neq \emptyset$, i.e., A is an 'open neighborhood' free cover, we consider three exhaustive cases. Case 1: $v \in R$. Since f is a satisfying assignment, every clause C_i contains a literal that is assigned the value 1 and a literal that is assigned the value 0. Hence, for all $v \in R$, v is adjacent to at least one vertex in the set A and at least one vertex in the set

B. Case 2: $v \in S \cup \overline{S}$. By assumption, each literal appears in at least one of the clauses. Hence, each vertex in set $S \cup \overline{S}$ is adjacent to at least one vertex in $R \subseteq A$. Also, by construction, each vertex in set $S \cup \overline{S}$ is adjacent to one vertex in $T \subseteq B$. Case 3: $v \in T$. By construction, each $v \in T$ is adjacent to a vertex $s_i \in S$ and $\overline{s_i} \in \overline{S}$ and thus has a neighbor in both sets A and B.
$\Longleftarrow$ Suppose now that the constructed graph G has an 'open neighborhood' free cover A, and let $B = V - A$. Define a truth assignment $f : U \longrightarrow \{0, 1\}$, such that $f(u_i) = 1$ if and only if $s_i \in A$. Since each vertex $t_i \in T$ is adjacent to only two vertices, $s_i \in S$ and $\overline{s_i} \in \overline{S}$, exactly one of these vertices must be in set A. Thus, for each literal u_i, $f(u_i) = 1$ if and only if $f(\overline{u_i}) = 0$, i.e., f is a legal assignment. Also, each vertex $r_i \in R$ has at least one vertex in A and one vertex in B and hence each clause C_i has at least one true literal and at least one false literal. Thus, f is a satisfying assignment. □

3.4 When X='is a dominating set'

Given a set $S \subset V$, a vertex v is called $k-$*dominated* by S, if it is dominated by (adjacent to) at least k vertices in S, i.e., $|N(v) \cap S| \geq k$. If every vertex in $V - S$ is $k-$dominated by S, then S is called a $k-$*dominating set*. A set T is called *total* $k-$*dominating set* if every vertex in V is $k-$dominated by T [7]. Note that, a $1-$dominating set is simply a dominating set, whereas a total $1-$dominating set is a total dominating set.

If X is the property of being a $k-$dominating set, then a set S is X free if and only if there is a vertex $v \in V - S$, such that, $|N(v) \cap S| < k$, i.e., there is a vertex $v \in V - S$ that is not dominated by set S. Thus we have the following result:

Theorem 15. *If X is the property of being a $k-$dominating set then a graph G has an X free cover if and only if there exists $\{u, v\} \subset V$, such that, $|N[u] \cap N[v]| \leq 2k - 2$.*

Proof. Let X be the property of being a $k-$dominating set. Set A is an X free cover if and only if both A and $V - A$ are X free if and only if there is a vertex $u \in V - A$ such that $|N(u) \cap A| < k$ and a vertex $v \in A$ such that $|N(u) \cap (V - A)| < k$ if and only if $|N[u] \cap N[v]| \leq 2k - 2$. □

Corollary 16. *If X is the property of being a dominating set then a graph G has an X free cover if and only if the diameter of G is greater than 2.*

If X is the property of being a total $k-$dominating set, then a set S is X free if and only if there is a vertex $v \in V$ such that $|N(v) \cap S| < k$.

Theorem 17. *If X is the property of being a total $k-$dominating set then a graph G has an X free cover if and only if there exists $\{u, v\} \subset V$ such that $|N(u) \cap N(v)| \leq 2k - 2$.*

Proof. Let X the property of being a total $k-$dominating set. Set A is an X free cover if and only if there are vertices u and v such that $|N(u) \cap A| < k$ and $|N(u) \cap (V - A)| < k$ if and only if $u = v$ and $\deg(u) \leq 2k - 2$, or $|N(u) \cap N(v)| \leq 2k - 2$. Also, if $\deg(u) \leq 2k - 2$ then for all $w \in V$, $|N(u) \cap N(w)| \leq 2k - 2$. □

A set S is called *distance $k-$dominating set* if every vertex in $V - S$ is within distance k of at least one vertex in S, i.e., for all $v \in V - S$, $d(v, S) \leq k$.

Theorem 18. *If X is the property of being a distance $k-$dominating set then a graph G has an X free cover if and only if the diameter of G is greater than $2k$.*

Proof. Let X be the property of being a total $k-$dominating set.

$\Longrightarrow$ Consider a graph G and let u and v be any two vertices with distance $d(u, v) > 2k$. Let $A = \{w | w \in V \wedge d(u, w) \leq k\}$. By construction, $u \in A$ and $v \in V - A$. Since $d(u, V - A) > k$, $V - A$ is X free. Also, since $d(u, v) > 2k$, the distance of v from every vertex in A is at least $k + 1$, i.e., $d(v, A) > k$. Hence, A is also X free.

$\Longleftarrow$ Let A be an X free cover in graph G. Then by the definitions of X and X free cover, there exist vertices $u \in A$ and $v \in V - A$, such that $d(u, V - A) > k$ and $d(v, A) > k$, which is possible only if $d(u, v) > 2k$. □

3.5 When X='is an alliance'

A vertex v in a set $A \subseteq V$ is said to be $k-$satisfied with respect to A if $|N(v) \cap A| \geq |N(v) \cap (V - A)| + k$, where $-\delta < k \leq \Delta$. A set A is a *defensive $k-$alliance* if all vertices in A are $k-$satisfied with respect to A [9]. A set S is defensive $k-$alliance free if and only if for every $T \subseteq S$ there is a vertex $v \in T$ such that $|N(v) \cap T| < |N(v) \cap (V - T)| + k$. Note that if $V = A \cup B$ is a vertex partition such that each vertex has as many or more neighbors outside the set in which it occurs than inside it, then for all $k > 0$, both A and B are defensive $k-$alliance free. Such a partition is called an *unfriendly partition* and one exists for every finite graph [2]. In addition, there exists a polynomial algorithm for finding an unfriendly partition of a given graph. Thus, for all $k > 0$, a defensive $k-$alliance free cover can be found in polynomial time. For $k = 0$, we have shown the following in [10].

Theorem 19. *A connected graph G has an alliance free cover if and only if G has a block that is other than an odd clique or an odd cycle.*

References

[1] D. Achlioptas, "The complexity of G-free colourability," Discrete Mathematics, 165-166, (1997), pp. 2130.

[2] R. Aharoni, E. C. Milner, and K. Prikry, “Unfriendly partitions of a graph,” Journal of Combinatorial Theory Series B, 50 (1990),pp 1-10.

[3] G. Chartrand, D. P. Geller, and S. Hedetniemi, “A generalization of the chromatic number,” in Proc. Cambridge Philosophical Society, 64 (1968), pp. 265-271.

[4] A. Farrugia, “Vertex partitioning into fixed additive induced hereditary properties,” preprint, 2004.

[5] M. R. Gary and D. S. Johnson, *Computers and intractability,* Freeman, NY, 1979.

[6] S. L. Hakimi and E. F. Schmeichel, “A note on the vertex arboricity of a graph,” SIAM J. Discrete Mathematics, 1 (1989), pp. 6467.

[7] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of domination in graphs,* Marcel Dekker, NY, 1998.

[8] O. Ore, *Theory of graphs,* AMS Colloquium Publications 38, American Mathematical Society, Providence, RI, 1962.

[9] K. H. Shafique and R. D. Dutton, “Maximum alliance-free and minimum alliance-cover sets,” Congressus Numerantium 162 (2003), pp. 139-146.

[10] K. H. Shafique and R. D. Dutton, “Partitioning a graph into alliance free sets,” preprint, 2004.