Maximum Alliance-Free and Minimum Alliance-Cover Sets

Khurram H. Shafique and Ronald D. Dutton
School of Computer Science
University of Central Florida
Orlando, FL USA 32816
khurram@cs.ucf.edu, dutton@cs.ucf.edu

Abstract

A *defensive* $k-$*alliance* in a graph $G = (V, E)$ is a set of vertices $A \subseteq V$ such that for every vertex $v \in A$, the number of neighbors v has in A is at least k more than the number of neighbors it has in $V - A$ (where k is the strength of defensive $k-$alliance). An *offensive* $k-$*alliance* is a set of vertices $A \subseteq V$ such that for every vertex $v \in \partial A$, the number of neighbors v has in A is at least k more than the number of neighbors it has in $V - A$ (where ∂A is the boundary of set A and is defined as $N[A] - A$). In this paper, we deal with two types of sets associated with these $k-$alliances: maximum $k-$alliance free and minimum $k-$alliance cover sets. Define a set $X \subseteq V$ to be *maximum* $k-$*alliance free* (for some type of $k-$alliance) if X does not contain any $k-$alliance (of that type) and is a largest such set. A set $Y \subseteq V$ is called *minimum* $k-$*alliance cover* (for some type of $k-$alliance) if Y contains at least one vertex from each $k-$alliance (of that type) and is a set of minimum cardinality satisfying this property. We present bounds on the cardinalities of maximum $k-$alliance free and minimum $k-$alliance cover sets and explore their inter-relation. The existence of forbidden subgraphs for graphs induced by these sets is also explored.

Keywords: Alliance, Defensive Alliance, Offensive Alliance, Alliance free Set, Alliance Cover Set, Cohesive Set.

1 Definitions and Notations

Alliances in graphs were first introduced by Hedetniemi, et. al.[3]. They proposed different types of alliances, namely (strong) defensive alliances, (strong) offensive alliances[1], global alliances[2], etc. In this paper, we consider generalizations of offensive and defensive alliances which we refer to as $k-$alliances, where the strength of an alliance is related to the value of parameter k.

Consider a graph $G = (V, E)$ without loops or multiple edges. A vertex v in set $A \subseteq V$ is said to be $k-$ satisfied with respect to A if $\deg_A(v) \geq \deg_{V-A}(v) + k$, where $\deg_A(v) = |N(v) \cap A| = |N_A(v)| = \deg(v) - \deg_{V-A}(v)$. A set A is a *defensive* $k-$ *alliance* if all vertices in A are $k-$satisfied with respect to A, where $-\Delta < k \leq \Delta$. Note that a defensive $(-1)-$alliance is a "defensive alliance" (as defined in [3]), and a defensive $0-$alliance is a "strong defensive alliance" or "cohesive set" [4]. Similarly, a set $A \subseteq V$ is an offensive $k-$alliance if $\forall v \in \partial A, \deg_A(v) \geq \deg_{V-A}(v)+k$, where $-\Delta+2 < k \leq \Delta$. Here, an offensive $1-$alliance is an "offensive alliance" and an offensive $2-$alliance is a "strong offensive alliance" (as defined in [1]).

A set $X \subseteq V$ is *defensive $k-$alliance free* ($k-$daf) if for all defensive $k-$alliances A, $A - X \neq \emptyset$, i.e., X does not contain any defensive $k-$alliance as a subset. A defensive $k-$alliance free set X is maximal if $\forall v \notin X$, $\exists S \subseteq X$ such that $S \cup \{v\}$ is a defensive $k-$alliance. A maximum $k-$daf set is a maximal $k-$daf set of largest cardinality. Let $\phi_k(G)$ be the cardinality of a maximum $k-$daf set of graph G. For simplicity of notation, we will refer to a maximum $k-$daf set of G as a $\phi_k(G)$-set. If a graph G does not have a defensive $k-$alliance (for some k), we say that $\phi_k(G)= |V(G)| = n$, for example, $\phi_k(P_n)= n, \forall k > 1$. Since $\forall k_1 \geq k_2$, a defensive k_2-alliance free set is also defensive k_1-alliance free, we have $\phi_{k_1}(G) \geq \phi_{k_2}(G)$ if and only if $k_1 \geq k_2$.

We define a set $Y \subseteq V$ to be a *defensive $k-$alliance cover* ($k-$dac) if for all defensive $k-$alliances A, $A \cap Y \neq \emptyset$, i.e., Y contains at least one vertex from each defensive $k-$alliance of G. A $k-$dac set Y is minimal if no proper subset of Y is a defensive $k-$alliance cover. A minimum $k-$dac set is a minimal cover of smallest cardinality. Let $\zeta_k(G)$ be the cardinality of a minimum $k-$dac set of graph G. Once again, we will refer to a minimum $k-$dac set of G as a $\zeta_k(G)$-set. When G does not have a defensive $k-$alliance (for some k), we say that $\zeta_k(G)= 0$.

For offensive $k-$alliances, we define two types of alliance free (cover) sets depending on whether or not the boundary vertices of an offensive alliance affect the definition of the set. A set $S \subseteq V$ is *offensive $k-$alliance free* ($k-$oaf) if for all offensive $k-$alliances A, $A - S \neq \emptyset$. S is *weak offensive $k-$alliance free* ($k-$woaf) if for all offensive $k-$alliances A, $N[A] - S \neq \emptyset$. Similarly, a set $T \subseteq V$ is an *offensive $k-$alliance cover* ($k-$oac) if for all offensive $k-$alliances A, $A \cap T \neq \emptyset$. T is a *weak offensive $k-$alliance cover* ($k-$woac) if for all offensive $k-$alliances A, $N[A] \cap T \neq \emptyset$. The maximum (weak) offensive $k-$alliance free sets and minimum (weak) offensive $k-$alliance cover sets are defined in the same fashion as their defensive counterparts. For a graph G, we define the following invariants

- $\phi_k(G)$ = Size of a maximum $k-$daf set of G
- $\zeta_k(G)$ = Size of a minimum $k-$dac set of G

- $\phi_k^o(G)$ = Size of a maximum $k-$oaf set of G
- $\zeta_k^o(G)$ = Size of a minimum $k-$oac set of G
- $\phi_k^w(G)$ = Size of a maximum $k-$woaf set of G
- $\zeta_k^w(G)$ = Size of a minimum $k-$woac set of G

In this paper, we explore the properties and bounds of the above defined invariants and their relationship with each other. In general we will refer to both offensive and defensive $k-$alliances as $k-$alliances. Similarly, the terms $k-$alliance free set and $k-$alliance cover set will encompass all types of alliance free sets and cover sets defined in this section. For other graph terminology and notation, we follow [6].

2 Basic Properties

Theorem 1. *$X \subseteq V$ is a $k-$alliance cover if and only if $V - X$ is $k-$alliance free.*

Proof. A set X is a defensive $k-$alliance free set if and only if, for every defensive $k-$alliance A, $A - X \neq \emptyset$ if and only if, for every defensive $k-$alliance A, $A \cap (V - X) \neq \emptyset$ if and only if $V - X$ is a defensive $k-$alliance cover.

The justification for the (weak) offensive alliance cover is similar. □

Corollary 2. $\phi_k(G) + \zeta_k(G) = \phi_k^o(G) + \zeta_k^o(G) = \phi_k^w(G) + \zeta_k^w(G) = n$

Corollary 3.

(i) *If V' is a minimal k-dac ($k-$oac) then, $\forall v \in V'$, there exists a defensive (offensive) $k-$alliance S_v for which $S_v \cap V' = \{v\}$.*

(ii) *If V' is a minimal k-wdac then, $\forall v \in V'$, there exists an offensive $k-$alliance S_v for which $N[S_v] \cap V' = \{v\}$.*

Since, $\forall k_1 > k_2$, a k_2-alliance free set is also a k_1-alliance free set and every k_1-oaf set is a k_1-woaf set, we have the following observation.

Observation 4. *For any graph G and $-\Delta < k_2 < k_1 \leq \Delta$,*

(i) $0 \leq \phi_{k_2}^o(G) \leq \phi_{k_1}^o(G) \leq \phi_{k_1}^w(G) \leq n$

(ii) $0 \leq \phi_{k_1}^w(G) \leq \phi_{k_2}^w(G) \leq n$

(iii) $0 \leq \phi_{k_2}(G) \leq \phi_{k_1}(G) \leq n$

Also note that every $k-$daf set X is a $k-$woaf set. Suppose not, then there is an offensive $k-$alliance A such that $N[A] \subseteq X$. Then $\forall v \in N[A]$, $\deg_{N[A]}(v) \geq \deg_{V-N[A]}(v) + k$, which implies that $N[A]$ is a defensive $k-$alliance and contradicts X being a $k-$daf set.

Observation 5. $\phi_k^w(G) \geq \phi_k(G)$

Suppose now a minimal k_1-dac set Y, $k_1 > -\delta(G)$, and let $A \subseteq Y$ such that A is an offensive k_2-alliance. Let $y \in A$, then by Corollary 3, there exists a defensive k_1-alliance S_y such that $S_y \cap Y = \{y\}$. Hence $\exists x \in \partial A - Y$ such that $\deg_A(x) \leq \deg_{V-A}(x) + 2 - k_1$. Also, since A is an offensive k_2-alliance, $\deg_A(x) \geq \deg_{V-A}(x) + k_2$. Combining the two inequalities, we get, $k_2 \leq 2 - k_1$. This leads to the following observation:

Observation 6. *For any graph G and every k_1, k_2 such that $k_1 > -\delta(G)$ and $k_2 > 2 - k_1$, $\phi_{k_2}^o(G) \geq \zeta_{k_1}(G)$*

3 Defensive $k-$Alliance Free & Cover Sets

For any k, such that $-\delta(G) < k \leq \Delta(G)$, we know that any independent set in a connected graph G is $k-$daf, therefore $\phi_k(G) \geq \beta_0(G)$, where $\beta_0(G)$ is the vertex independence number of graph G. We can further improve this bound by noting that the addition of any $\left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$ vertices to an independent set will not produce a defensive $k-$alliance in the new set, hence, $\phi_k(G) \geq \beta_0(G) + \left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1$. Since, every $A \subset V$, such that $|A| \geq n - \left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil$, is a defensive $k-$alliance, $\phi_k(G) < n - \left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil$.

Observation 7. *If G is a connected graph and $-\delta(G) < k \leq \Delta(G)$ then*

$$\beta_0(G) + \left\lceil \frac{\delta(G)}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 \leq \phi_k(G) < n - \left\lfloor \frac{\delta(G)}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil$$

Next we present the values of $\phi_k(G)$ for some common graph families.

Observation 8. *If G is an Eulerian graph and $-\frac{\delta(G)}{2} < i \leq \frac{\Delta(G)}{2}$, then $\phi_{2i-1}(G) = \phi_{2i}(G)$.*

Observation 9. *For the complete graph K_n and $-n+1 < k < n$,*

$$\phi_k(K_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{k}{2} \right\rceil & \text{for odd } n \\ \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor & \text{for even } n. \end{cases}$$

Observation 10. *For the complete bipartite graph $K_{p,q}$, where $p \leq q$ and $-p < k \leq p$,*

$$\phi_k(K_{p,q}) = \begin{cases} q + \left\lceil \frac{p}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor - 1 & \text{for odd } p \\ \\ q + \left\lceil \frac{p}{2} \right\rceil + \left\lceil \frac{k}{2} \right\rceil - 1 & \text{for even } p. \end{cases}$$

Note that the upper and lower bounds of Observation 7 coincide for both K_n and $K_{p,q}$, when k is even. We have shown in [5] that the following lower bound holds for $\phi_k(G)$.

Theorem 11. *For every connected graph G and $0 \leq k \leq \Delta$,*

$$\phi_k(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$$

.

We believe (but have been unable to prove) the following extension of the above theorem:

Conjecture 1. *If G is a connected graph and $-\delta(G) < k \leq \Delta(G)$ then*

$$\phi_k(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{k}{2} \right\rfloor$$

Next, we show that no forbidden subgraph characterization exists for the graphs induced by minimal $k-$dac sets.

Theorem 12. *Let G be any graph and r an integer such that $r \geq 2$. Then, for all $k \geq 2 - r$, there is a graph G', such that G' contains G as an induced subgraph and $\zeta_k(G') = r$.*

Proof. Let a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and construct a graph $G' = (V', E')$ as follows: $V' = V \cup X \cup Y$, where $X = \left\{x_i^j,\ 1 \leq i \leq n,\right.$ $\left. 1 \leq j \leq \max(2r + k, \Delta(G) - k + 1)\right\}$ and $Y = \{y_1, y_2, \ldots, y_{2r+k-2}\}$. $E' = E \cup E_1 \cup E_2$, where $E_1 = \left\{v_i x_i^j,\ v_i \in V,\ x_i^j \in X\right\}$ and $E_2 = \left\{x_i^j y_l,\ x_i^j \in X,\ y_l \in Y\right\}$. Thus $\delta(G') = 2r + k - 1$. Since by Observation 7, $\zeta_k(G') \geq \left\lfloor \frac{\delta(G')}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 1$, we have $\zeta_k(G') \geq \left\lfloor \frac{2r+k-1}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil + 1 = r$.

Now consider $C \subseteq Y$ such that $|C| = r$. We claim that C is a $k-$dac set of graph G'. Suppose not. Then there exists a defensive $k-$alliance $S \subseteq V' - C$ in G'. Let $v \in S$. Since $\forall x \in X,\ \deg(x) = 2r + k - 1$, if $v \in X$ then $\deg_S(v) \leq r + k - 1 < \deg_C(v) + k = r + k$, which is contrary to S being a defensive $k-$alliance. Hence $S \cap X = \emptyset$. Now let $v \in V$. By construction of graph G', $\forall v \in V,\ \deg_X(v) + k \geq \Delta(G) + 1 > \deg_{V'-X}(v) \geq \deg_S(v)$, again a contradiction. The only remaining case is $S \subset Y$, which is not possible as $\forall v \in S,\ \deg_S(v) = 0 < deg_{V'-S}(y) + k \leq n(2r + k) + k$. Hence $S = \emptyset$ and C is a $k-$dac set. Thus $\zeta_k(G') \leq r$.

Combining the two results, we get $\zeta_k(G') = r$. □

4 Offensive $k-$Alliance Free & Cover Sets

In this section, we study the properties of the free sets and cover sets associated with offensive $k-$alliances. We begin by presenting the values of $\phi_k^o(G)$ and $\phi_k^w(G)$ for some special classes of graphs.

Observation 13. *For the complete graph K_n, and $-n+3 < k < n$*

$$\phi_k^o(K_n) = \phi_k(K_n) - 1 = \left\lfloor \frac{n+k}{2} \right\rfloor - 1$$

$$\phi_k^w(K_n) = n - 1$$

Observation 14. *For the complete bipartite graph $K_{p,q}$, $p \leq q$, and $-p+2 < k \leq q$*

$$\phi_k^o(K_{p,q}) = \begin{cases} \lceil \frac{q}{2} \rceil + \lceil \frac{p}{2} \rceil + 2 \lfloor \frac{k}{2} \rfloor - 2 & p \text{ \& } q \text{ both odd} \\ \lceil \frac{q}{2} \rceil + \lceil \frac{p}{2} \rceil + 2 \lceil \frac{k}{2} \rceil - 2 & p \text{ \& } q \text{ both even} \\ \lceil \frac{q}{2} \rceil + \lceil \frac{p}{2} \rceil + k - 2 & \text{otherwise} \end{cases}$$

$$\phi_k^w(K_{p,q}) = n - 2, \; p, q \neq 1$$

It is interesting to note that while complete graphs attain the lower bound for $\phi_k(G)$, they have the maximum value for $\phi_k^w(G)$.

Lemma 15. *If S is an offensive k_1- alliance then*

(i) *for all offensive k_2-alliances $S' \subseteq V - S$ such that $k_1+k_2 > 0$, $\partial S \cap \partial S' = \emptyset$.*

(ii) *for all defensive k_2-alliances $S' \subseteq V - S$ such that $k_1+k_2 > 0$, $\partial S \cap S' = \emptyset$.*

Theorem 16. *For a connected graph G, if X is a maximal k_1-woaf set and $Y = V - X$ then*

(i) *$\forall k_2 > -k_1$, Y is a k_2-woaf set (and hence, X is a k_2-woac set), and*

(ii) *$\forall k_2 > \max(-k_1, -\delta(G))$, Y is a k_2-daf set (hence, X is a k_2-dac set).*

Proof. For i), let $k_2 > -k_1$ and suppose there exists an offensive k_2-alliance S for which $N[S] \subseteq Y$. Let $x \in \partial S$. From Corollary 3, there is an offensive k_1-alliance S_x for which $N[S_x] \cap Y = \{x\}$. If $x \in \partial S_x$, then from Lemma 15, S and S_x cannot be disjoint, a contradiction. So we must assume that $x \in S_x$. But then, $N(x) \subseteq \partial S_x \subseteq X$, which leads to a contradiction since x must have at least one neighbor in $S \subseteq Y$. Thus, Y is a k_2-woaf set and, from Theorem 1, X is a k_2-woac set.

For ii), let $k_2 > \max(-k_1, -\delta(G))$ and suppose there exists a defensive k_2-alliance $S \subseteq Y$. Let $x \in S$. From Corollary 3, there exists an offensive k_1-alliance S_x for which $N[S_x] \cap Y = \{x\}$. If $x \in \partial S_x$ then from Lemma 15, S and S_x cannot be disjoint, a contradiction. So we must assume that $x \in S_x$, but then $N(x) \subseteq \partial S_x \subseteq X$, which is not possible since $\deg_S(x) \geq (deg(x)+k_2)/2 > 0$. Hence, Y is a k_2-daf set and, from Theorem 1, X is a k_2-dac set. □

Corollary 17.

(i) *Every maximal k_1-woaf set contains a minimal k_2-woac set, $\forall k_2 > -k_1$.*

(ii) *Every maximal k_1-woaf set contains a minimal k_2-dac set, $\forall k_2 > \max(-k_1, -\delta(G))$.*

Since every $k-$woaf is also $l-$woaf $\forall l > k$, by Theorem 1, every $k-$woac is also $l-$woac. This observation leads to the following corollary of Theorem 16.

Corollary 18. $\forall k > 0,\ \zeta_k^w(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$

It is easy to prove that $\forall k \geq 0,\ \zeta_k^w(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $G \approx K_2$ and $k < 2$.

We conclude this section by presenting a result for $\zeta_k^w(G)$ similar to the one for $\zeta_k(G)$ in Theorem 12.

Theorem 19. *Let G be any graph and r an integer such that $r \geq 1$. Then there is a graph G' with $\zeta_k^w(G') = r$, which contains G as an induced subgraph.*

Proof. Let a graph $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ and construct a graph $G' = (V', E')$ as follows: $V' = V \cup X \cup Y$, where $X = \{x_1, x_2, \ldots, x_r\}$ and Y is the union of disjoint sets $Y_1, Y_2, \ldots, Y_r$, such that $\forall i,\ |Y_i| = n - k + 1$. $E' = E \cup E_1 \cup E_2 \cup E_3$, where $E_1 = \{v_i x_j,\ v_i \in V,\ x_j \in X\}$, $E_2 = \bigcup_{i=1}^{r} \{x_i y,\ \forall y \in Y_i\}$ and $E_3 = \{yz|\ y, z \in Y_i,\ 1 \leq i \leq r\}$. Hence, G' is obtained by adding r vertex disjoint cliques $Y_i \cup \{x_i\}$, each of order $n - k + 2$ vertices and making each x_i adjacent to every vertex of V.

It is easy to see that X is a $k-$woac set of graph G', i.e. $\zeta_k^w(G') \leq |X| = r$. We claim that $\zeta_k^w(G') = r$. Suppose not and let $C \subset V$ be a $k-$woac set of graph G' such that $|C| < r$. By pigeon hole principle, there exists Y_i such that $(Y_i \cup \{x_i\}) \cap C = \emptyset$. Since $\partial Y_i = \{x_i\}$ and $deg_{Y_i}(x_i) = n+k+1 > deg_{V'-Y_i}(x_i) + k = n + k$, Y_i is an offensive $k-$alliance in G' such that $N[Y_i] \subseteq V' - C$, which is contrary to C being a $k-$woac set of graph G'. Hence $\zeta_k^w(G') \geq r$.

Combining the two results, we get $\zeta_k^w(G') = r$. □

5 Open Problems

1. Determine the computational complexity of finding each of $\phi_k(G)$, $\phi_k^o(G)$ and $\phi_k^w(G)$.

2. Find efficient algorithms for computing $\phi_k(G)$, $\phi_k^o(G)$ and $\phi_k^w(G)$.
3. Determine tight upper and lower bounds for $\phi_k^o(G)$ and $\phi_k^w(G)$ and characterize extremal graphs.
4. Determine the values of $\phi_k(G)$, $\phi_k^o(G)$ and $\phi_k^w(G)$ for other classes of graphs, for example, grid graphs.
5. Do results similar to Theorem 12 and Theorem 19 hold for $\zeta_k^o(G)$?
6. Study the cover and free sets of other alliances, e.g., dual alliances and global alliances, and their relationship.

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