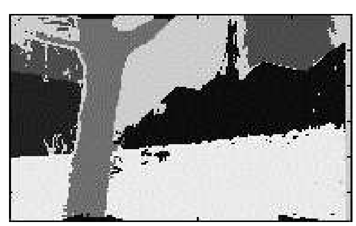
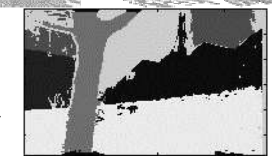
Graph Theoretical Techniques for Image Segmentation

Region Segmentation



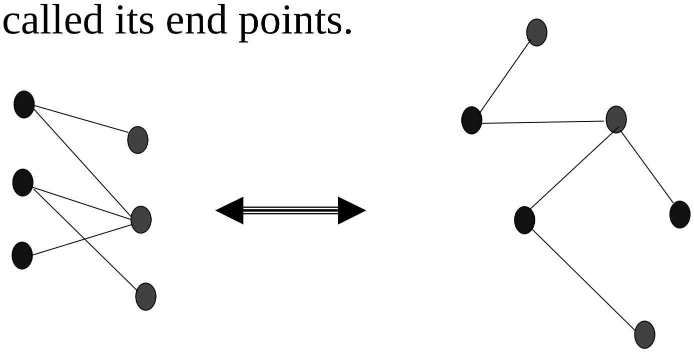
Region Segmentation



- Find sets of pixels, $R_1, R_2, \ldots, R_n$ such that
  - $\bigcup_i R_i = I$
  - $\forall i \neq j, R_i \cap R_j = \varnothing$
  - All pixels in region *i* satisfy some constraint of similarity.

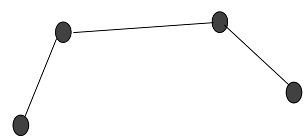
Graph

- A graph G(V,E) is a triple consisting of a vertex set V(G) an edge set E(G) and a relation that associates with each edge two vertices called its end points.



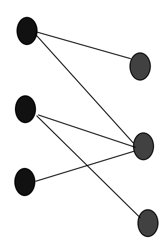
Path

- A path is a sequence of edges e1, e2, e3, … en. Such that each (for each i>2 & i<n) edge ei is adjacent to e(i+1) and e(i-1). e1 is only adjacent to e2 and en is only adjacent to e(n-1)

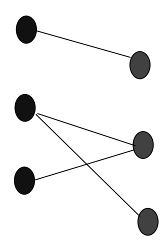


Connected & Disconnected Graph

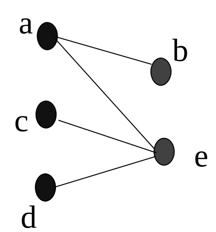
- A graph *G* is connected if there is a path from every vertex to every other vertex in *G*.



- A graph *G* that is not connected is called disconnected graph.



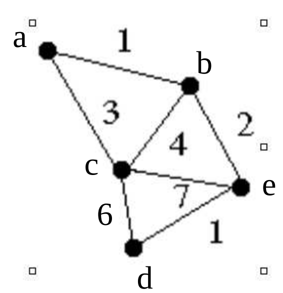
Graphs Representations



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency Matrix: W

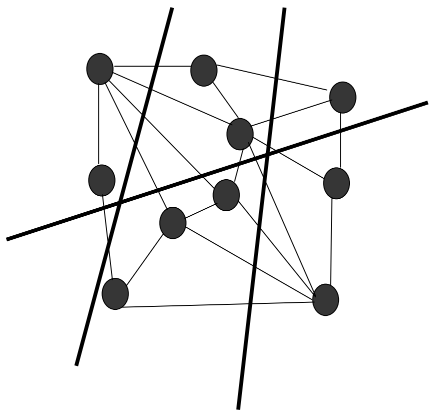
Weighted Graphs and Their Representations



$$\begin{bmatrix} 0 & 1 & 3 & \infty & \infty \\ 1 & 0 & 4 & \infty & 2 \\ 3 & 4 & 0 & 6 & 7 \\ \infty & \infty & 6 & 0 & 1 \\ \infty & 2 & 7 & 1 & 0 \end{bmatrix}$$

Weight Matrix: W

Minimum Cut

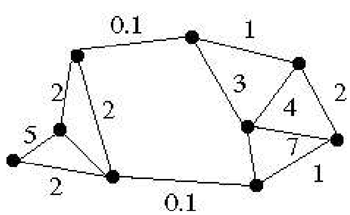


A cut of a graph *G* is the set of edges *S* such that removal of *S* from *G* disconnects *G*.

Minimum cut is the cut of minimum weight, where weight of cut <A,B> is given as

$$w(\langle A,B\rangle)=\sum_{x\in A, y\in B} w(x,y)$$

Minimum Cut and Clustering



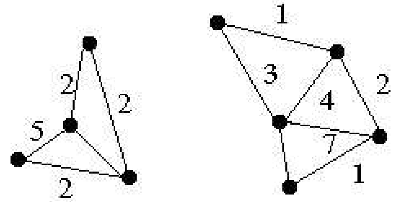
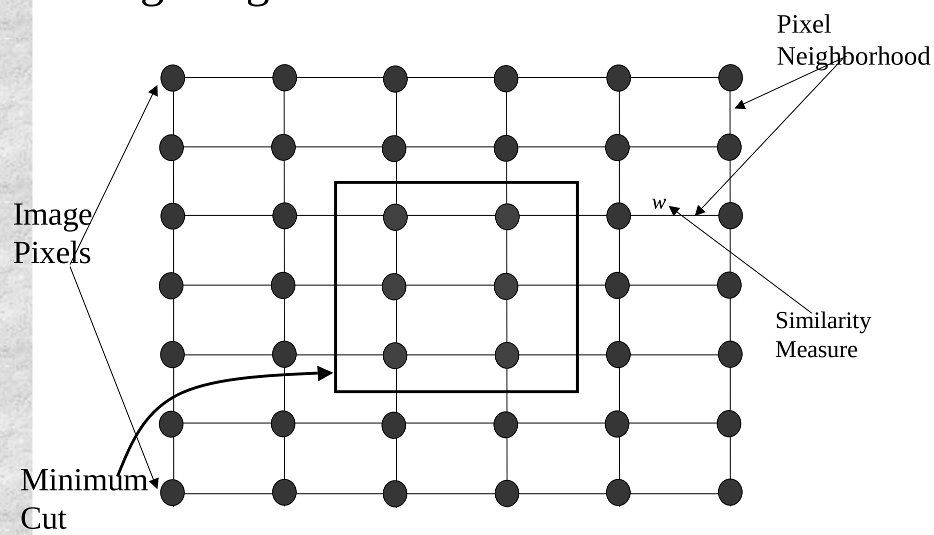
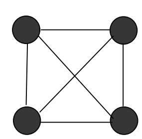


Image Segmentation & Minimum Cut



Minimum Cut

- There can be more than one minimum cut in a given graph

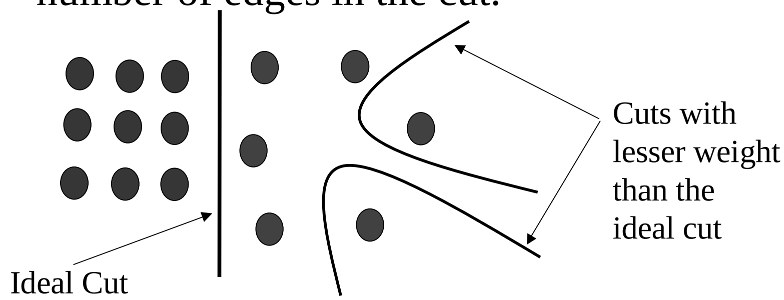


- All minimum cuts of a graph can be found in polynomial time[1].

[1]H. Nagamochi, K. Nishimura and T. Ibaraki, “Computing all small cuts in an undirected network. SIAM J. Discrete Math. 10 (1997) 469-481.

Drawbacks of Minimum Cut

- Weight of cut is directly proportional to the number of edges in the cut.



Normalized Cuts[1]

- Normalized cut is defined as

$$N_{cut}(A,B) = \frac{w(\langle A,B \rangle)}{\sum_{x \in A, y \in V} w(x,y)} + \frac{w(\langle A,B \rangle)}{\sum_{z \in B, y \in V} w(z,y)}$$

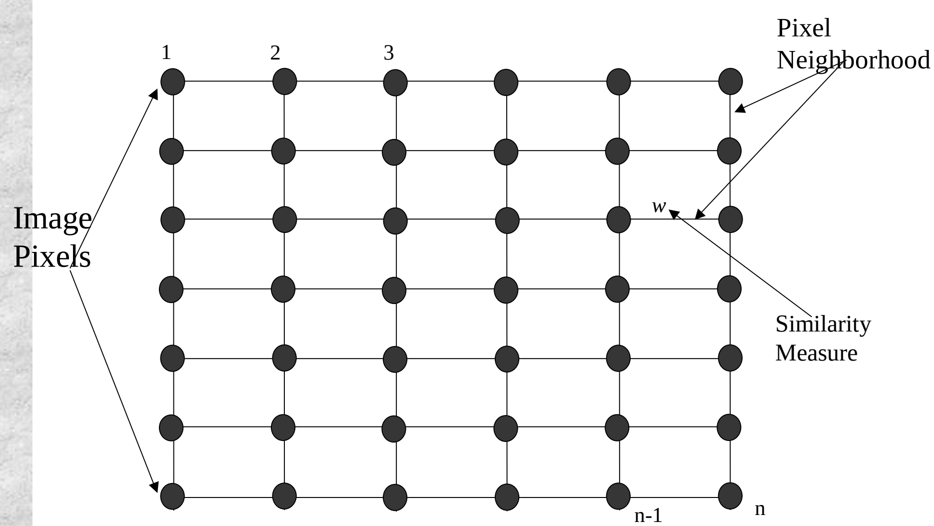
- $N_{cut}(A,B)$ is the measure of dissimilarity of sets A and B.
- Minimizing $N_{cut}(A,B)$ maximizes a measure of similarity within the sets A and B

[1]J. Shi and J. Malik, "Normalized Cuts & Image Segmentation," IEEE Trans. of PAMI, Aug 2000.

Finding Minimum Normalized-Cut

- Finding the Minimum Normalized-Cut is NP-Hard.
- Polynomial Approximations are generally used for segmentation

Finding Minimum Normalized-Cut



Finding Minimum Normalized-Cut

$W = N \times N$ symmetric matrix, where

$$W(i,j) = \begin{cases} e^{-\|F_i - F_j\| / \sigma_F^2} \times e^{-\|X_i - X_j\| / \sigma_X^2} & \text{if } j \in N(i) \\ 0 & \text{otherwise} \end{cases}$$

$\|F_i - F_j\| =$ Image feature similarity

$\|X_i - X_j\| =$ Spatial Proximity

$D = N \times N$ diagonal matrix, where $D(i,i) = \sum_j W(i,j)$

Finding Minimum Normalized-Cut

- It can be shown that $\min N_{cut} = \min_{\mathbf{y}} \frac{\mathbf{y^T}(\mathbf{D}-\mathbf{W})\mathbf{y}}{\mathbf{y^T D y}}$

such that $y(i) \in \{1, -b\}$, $0 < b \le 1$, and $\mathbf{y}^T \mathbf{D1} = 0$

- If y is allowed to take real values then the minimization can be done by solving the generalized eigenvalue system

$$(\mathbf{D} - \mathbf{W})\mathbf{y} = \lambda \mathbf{D y}$$

Algorithm

- Compute matrices W & D
- Solve $(\mathbf{D} - \mathbf{W})\mathbf{y} = \lambda \mathbf{D}\mathbf{y}$ for eigen vectors with the smallest eigen values
- Use the eigen vector with second smallest eigen value to bipartition the graph
- Recursively partition the segmented parts if necessary.

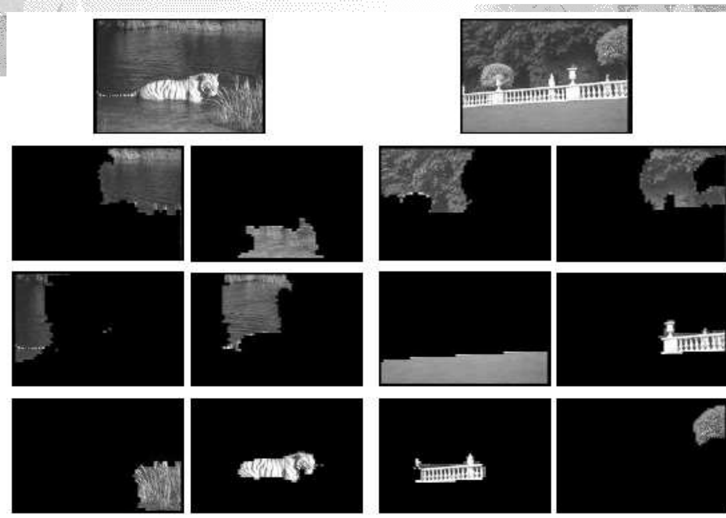
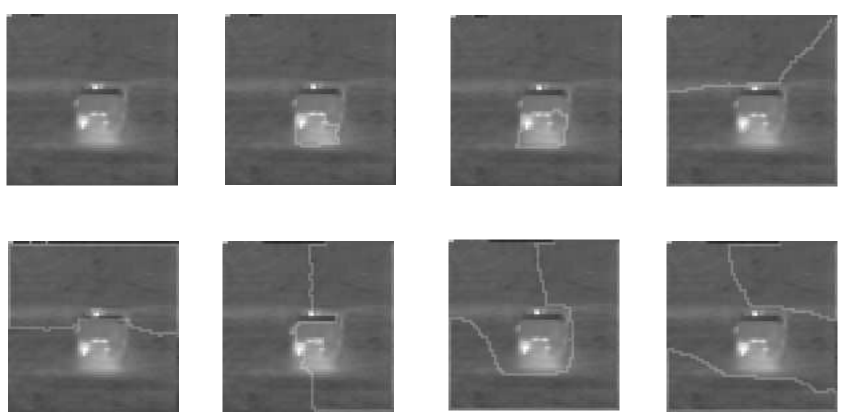
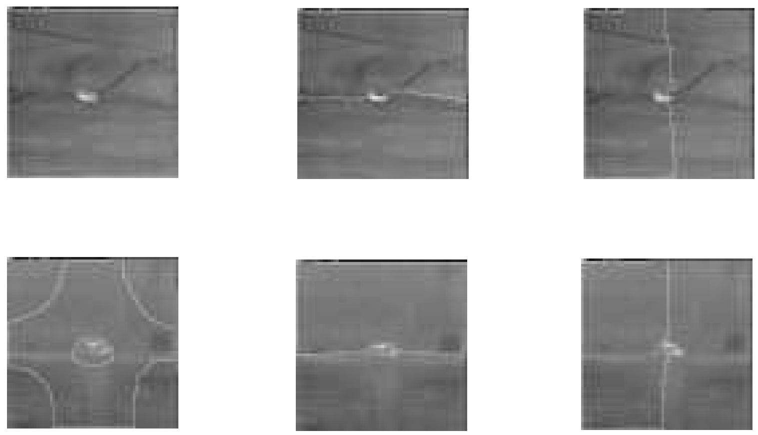


Figure from “Image and video segmentation: the normalised cut framework”, by Shi and Malik, 1998

F igure from “Normalized cuts and image segmentation,” Shi and Malik, 2000





Drawbacks of Minimum Normalized Cut

- Huge Storage Requirement and time complexity
- Bias towards partitioning into equal segments
- Have problems with textured backgrounds

Suggested Reading

- Chapter 14, David A. Forsyth and Jean Ponce, “Computer Vision: A Modern Approach”.
- Jianbo Shi, Jitendra Malik, “Normalized Cuts and Image Segmentation,” IEEE Transactions on Pattern Analysis and Machine Intelligence, 1997