Solution for Assignment #2

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Prove the following results

1. \( g_{\sigma_1}(x) \ast (g_{\sigma_2}(x) \ast I) = g_{\sigma}(x) \ast I \). Also, find the value of \( \sigma \) in terms of \( \sigma_1 \) and \( \sigma_2 \).

Proof:

\[
L = g_{\sigma_1}(x) \ast (g_{\sigma_2}(x) \ast I) \\
\Rightarrow F(L) = F(g_{\sigma_1}(x) \ast (g_{\sigma_2}(x) \ast I)) = F(g_{\sigma_1}(x)) F((g_{\sigma_2}(x) \ast I)) \\
= \frac{\sqrt{2\pi}}{\sigma_1} g_{\frac{1}{\sqrt{2\pi}}} (u) \frac{\sqrt{2\pi}}{\sigma_2} g_{\frac{1}{\sqrt{2\pi}}} (u) F(I) \\
= \left[ \frac{\sqrt{2\pi}}{\sigma_1} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2\sigma_1^2}{2} \right) \right] \left[ \frac{\sqrt{2\pi}}{\sigma_2} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2\sigma_2^2}{2} \right) \right] F(I) \\
= \exp \left( -\frac{u^2(\sigma_1^2 + \sigma_2^2)}{2} \right) F(I) \\
= \frac{\sqrt{2\pi}}{\sigma} \left[ \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{u^2\sigma^2}{2} \right) \right] F(I) \\
= F(g_{\sigma}(x)) F(I) \\
\Rightarrow F(L) = F(g_{\sigma}(x) \ast I) \\
\Rightarrow L = g_{\sigma}(x) \ast I
\]

where \( \sigma = \sqrt{\sigma_1^2 + \sigma_2^2} \)

2. \( g_{\sigma}(x, y) = g_{\sigma}(x) g_{\sigma}(y) \).

Proof:

\[
g_{\sigma}(x, y) = \frac{1}{2\pi\sigma^2} \exp \left( -\frac{x^2+y^2}{2\sigma^2} \right) \\
= \left( \frac{1}{\sqrt{2\pi}\sigma} \right) \left( \frac{1}{\sqrt{2\pi}\sigma} \right) \exp \left( -\frac{x^2}{2\sigma^2} \right) \exp \left( -\frac{y^2}{2\sigma^2} \right) \\
= \left[ \left( \frac{1}{\sqrt{2\pi}\sigma} \right) \exp \left( -\frac{x^2}{2\sigma^2} \right) \right] \left[ \left( \frac{1}{\sqrt{2\pi}\sigma} \right) \exp \left( -\frac{y^2}{2\sigma^2} \right) \right] \\
= g_{\sigma}(x) g_{\sigma}(y)
\]
3. \( g_\sigma(x,y) \ast I = g_\sigma(x) \ast (g_\sigma(y) \ast I) \).

Proof:

Method 1.

\[
L = g_\sigma(x,y) \ast I \\
\Rightarrow F(L) = F(g_\sigma(x,y) \ast I) = F(g_\sigma(x,y)) F(I) \quad \text{By Convolution Theorem}
\]

\[
= \frac{\partial}{\partial x} g_\frac{1}{2}(u,v) F(I) \quad \text{By Problem 2}
\]

\[
= \left( \frac{\sqrt{\pi}}{\sigma} \right) \left( \frac{\sqrt{\pi}}{\sigma} \right) g_\frac{1}{2}(u) g_\frac{1}{2}(v) F(I)
\]

\[
= F(g_\sigma(x)) F(g_\sigma(y)) F(I)
\]

\[
\Rightarrow F(L) = F(g_\sigma(x) \ast (g_\sigma(y) \ast I))
\]

\[
\Rightarrow L = g_\sigma(x) \ast (g_\sigma(y) \ast I)
\]

Method 2.

\[
g_\sigma(x,y) \ast I = \int \int g_\sigma(x-x',y-y') I(x',y') dy'dx'
\]

\[
= \int \int g_\sigma(x-x') g_\sigma(y-y') I(x',y') dy'dx' \quad \text{By Problem 2}
\]

\[
= \int g_\sigma(x-x') \left[ \int g_\sigma(y-y') I(x',y') dy' \right] dx'
\]

\[
= \int g_\sigma(x-x') (g_\sigma(y) \ast I) (x',y') dx'
\]

\[
= g_\sigma(x) \ast (g_\sigma(y) \ast I)
\]

4. \( \Delta (g_\sigma(x,y) \ast I) = (\Delta g_\sigma(x,y)) \ast I \).

Proof:

\[
L = \frac{\partial}{\partial x} (g_\sigma(x,y) \ast I)
\]

\[
\Rightarrow F(L) = F \left( \frac{\partial}{\partial x} (g_\sigma(x,y) \ast I) \right) = uF(g_\sigma(x,y) \ast I) \quad \text{Since } F \left( \frac{\partial}{\partial x} f \right) = uF(f)
\]

\[
= uF(g_\sigma(x,y)) F(I) \quad \text{By Convolution Theorem}
\]

\[
\Rightarrow F(L) = F \left[ \left( \frac{\partial}{\partial x} g_\sigma(x,y) \right) \ast I \right]
\]

\[
\Rightarrow L = \left( \frac{\partial}{\partial x} g_\sigma(x,y) \right) \ast I
\]

Similarly, we can prove that \( \frac{\partial}{\partial y} (g_\sigma(x,y) \ast I) = \left( \frac{\partial}{\partial y} g_\sigma(x,y) \right) \ast I \).

Hence, \( \Delta (g_\sigma(x,y) \ast I) = \left[ \frac{\partial}{\partial x} (g_\sigma(x,y) \ast I) \frac{\partial}{\partial y} (g_\sigma(x,y) \ast I) \right] T = \left[ (\frac{\partial}{\partial x} g_\sigma(x,y)) \ast I (\frac{\partial}{\partial y} g_\sigma(x,y)) \ast I \right] T = \left[ (\frac{\partial}{\partial x} g_\sigma(x,y)) \frac{\partial}{\partial y} g_\sigma(x,y) \right] T I = (\Delta g_\sigma(x,y)) \ast I \)
5. $|\Delta|$ is isotropic (rotation invariant).

**Proof:** Let $I(x, y)$ be an image and $I(x', y')$ be the rotated image such that $[x' y']^T = R[x y]^T$ where $R$ is a 2D rotation matrix. Since $R^{-1} = R^T$, we have, $[x y]^T = R^T[x y']^T$. Hence,

$$x = x' \cos \theta + y' \sin \theta \quad y = -x' \sin \theta + y' \cos \theta$$

Now the derivatives of $I$ in rotated coordinate system are given by

$$\frac{\partial I}{\partial x'} = \frac{\partial I}{\partial x} \cos \theta - \frac{\partial I}{\partial y} \sin \theta$$
$$\frac{\partial I}{\partial y'} = \frac{\partial I}{\partial x} \sin \theta + \frac{\partial I}{\partial y} \cos \theta$$

The magnitude of gradient $|\Delta|$ in rotated coordinate system is given by

$$|\Delta|(x', y') = \sqrt{\left(\frac{\partial I}{\partial x'}\right)^2 + \left(\frac{\partial I}{\partial y'}\right)^2}$$
$$= \sqrt{\left(\frac{\partial I}{\partial x} \cos \theta - \frac{\partial I}{\partial y} \sin \theta\right)^2 + \left(\frac{\partial I}{\partial x} \sin \theta + \frac{\partial I}{\partial y} \cos \theta\right)^2}$$
$$= \sqrt{\left(\frac{\partial I}{\partial x} \cos \theta\right)^2 - 2\frac{\partial I}{\partial x} \frac{\partial I}{\partial y} \cos \theta \sin \theta + \left(\frac{\partial I}{\partial y} \sin \theta\right)^2 + \left(\frac{\partial I}{\partial x} \sin \theta\right)^2 + \left(\frac{\partial I}{\partial y} \cos \theta\right)^2}$$
$$= \sqrt{\left(\frac{\partial I}{\partial x}\right)^2 + \left(\frac{\partial I}{\partial y}\right)^2} (\cos^2 \theta + \sin^2 \theta)$$

$$= |\Delta|(x, y)$$

6. $\Delta^2$ is linear and isotropic (rotation invariant).

**Proof:**

**Linearity** Let $f(x, y)$ and $g(x, y)$ be two functions and $k$ be any scalar, then we have,

$$\Delta^2 (f + g) = \frac{\partial^2}{\partial x^2} (f + g) + \frac{\partial^2}{\partial y^2} (f + g)$$
$$= \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial x^2} g + \frac{\partial^2}{\partial y^2} f + \frac{\partial^2}{\partial y^2} g$$

By linearity of derivative

$$= \left[\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f\right] + \left[\frac{\partial^2}{\partial x^2} g + \frac{\partial^2}{\partial y^2} g\right]$$

$$= \Delta^2 f + \Delta^2 g$$

Also,

$$\Delta^2 (kf) = \frac{\partial^2}{\partial x^2} (kf) + \frac{\partial^2}{\partial y^2} (kf)$$
$$= k\frac{\partial^2}{\partial x^2} f + k\frac{\partial^2}{\partial y^2} f$$

By linearity of derivative

$$= k\left[\frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f\right]$$
$$= k\Delta^2 f$$

Hence, $\Delta^2$ is linear.

**Rotational Invariance**

Let $I(x, y)$ be an image and $I(x', y')$ be the rotated image such that $[x' y']^T = R[x y]^T$ where $R$ is a 2D rotation matrix. Since $R^{-1} = R^T$, we have, $[x y]^T = R^T[x y']^T$. Hence,

$$x = x' \cos \theta + y' \sin \theta \quad y = -x' \sin \theta + y' \cos \theta$$
Finally, the Laplacian of the image in rotated coordinate system is given by

\[ I' = \frac{\partial I}{\partial x'} = \frac{\partial I}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial I}{\partial y} \frac{\partial y}{\partial x'} = \frac{\partial I}{\partial x} \cos \theta - \frac{\partial I}{\partial y} \sin \theta \]

\[ I'_y = \frac{\partial I}{\partial y'} = \frac{\partial I}{\partial x} \frac{\partial x}{\partial y'} + \frac{\partial I}{\partial y} \frac{\partial y}{\partial y'} = \frac{\partial I}{\partial x} \sin \theta + \frac{\partial I}{\partial y} \cos \theta \]

Similarly the second derivatives of \( I \) in rotated coordinate system are computed as follows:

\[ \frac{\partial I^2}{\partial x^2} = \frac{\partial I}{\partial x} \frac{\partial I}{\partial x} + \frac{\partial I}{\partial y} \frac{\partial I}{\partial y} \]

\[ = \frac{\partial}{\partial x} \left[ \frac{\partial I}{\partial x} \cos \theta - \frac{\partial I}{\partial y} \sin \theta \right] \cos \theta - \frac{\partial}{\partial y} \left[ \frac{\partial I}{\partial x} \cos \theta - \frac{\partial I}{\partial y} \sin \theta \right] \sin \theta \]

\[ = \left[ \frac{\partial^2 I}{\partial x^2} \cos \theta - \frac{\partial^2 I}{\partial y \partial x} \sin \theta \right] \cos \theta - \left[ \frac{\partial^2 I}{\partial x \partial y} \cos \theta - \frac{\partial^2 I}{\partial y^2} \sin \theta \right] \sin \theta \]

\[ = \frac{\partial^2 I}{\partial x^2} \cos^2 \theta - \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta - \frac{\partial^2 I}{\partial y^2} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \sin^2 \theta \]

\[ = \frac{\partial^2 I}{\partial x^2} \sin^2 \theta - 2 \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \cos^2 \theta \]

\[ \frac{\partial I^2}{\partial y^2} = \frac{\partial I}{\partial x} \frac{\partial I}{\partial x} + \frac{\partial I}{\partial y} \frac{\partial I}{\partial y} \]

\[ = \frac{\partial}{\partial x} \left[ \frac{\partial I}{\partial x} \sin \theta + \frac{\partial I}{\partial y} \cos \theta \right] \sin \theta + \frac{\partial}{\partial y} \left[ \frac{\partial I}{\partial x} \sin \theta + \frac{\partial I}{\partial y} \cos \theta \right] \cos \theta \]

\[ = \left[ \frac{\partial^2 I}{\partial x^2} \sin \theta + \frac{\partial^2 I}{\partial x \partial y} \cos \theta \right] \sin \theta + \left[ \frac{\partial^2 I}{\partial x \partial y} \sin \theta + \frac{\partial^2 I}{\partial y^2} \cos \theta \right] \cos \theta \]

\[ = \frac{\partial^2 I}{\partial x^2} \sin^2 \theta + \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \cos^2 \theta \]

\[ = \frac{\partial^2 I}{\partial x^2} \sin^2 \theta + 2 \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \cos^2 \theta \]

Finally, the Laplacian of the image in rotated coordinate system is given by

\[ \Delta^2 I(x', y') = \frac{\partial I^2}{\partial x^2} + \frac{\partial I^2}{\partial y^2} \]

\[ = \frac{\partial^2 I}{\partial x^2} \cos^2 \theta - 2 \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \sin^2 \theta \]

\[ + \left[ \frac{\partial^2 I}{\partial x^2} \sin^2 \theta + 2 \frac{\partial^2 I}{\partial x \partial y} \sin \theta \cos \theta + \frac{\partial^2 I}{\partial y^2} \cos^2 \theta \right] \]

\[ = \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \]

\[ = \Delta^2 I(x, y) \]
7. \( \Delta^2 (g_\sigma(x, y) * I) = (\Delta^2 g_\sigma(x, y)) * I \).

Proof:

\[
L = \Delta^2 (g_\sigma(x, y) * I) \\
\Rightarrow F(L) &= F(\Delta^2 (g_\sigma(x, y) * I)) \\
&= F\left( \frac{\partial^2}{\partial x^2} (g_\sigma(x, y) * I) + \frac{\partial^2}{\partial y^2} (g_\sigma(x, y) * I) \right) \\
&= F\left( \frac{\partial^2}{\partial x^2} (g_\sigma(x, y) * I) \right) + F\left( \frac{\partial^2}{\partial y^2} (g_\sigma(x, y) * I) \right) \\
&= u^2 F(g_\sigma(x, y) * I) + v^2 F(g_\sigma(x, y) * I) \\
&= u^2 F(g_\sigma(x, y)) F(I) + v^2 F(g_\sigma(x, y)) F(I) \\
&= [u^2 F(g_\sigma(x, y))] F(I) + [v^2 F(g_\sigma(x, y))] F(I) \\
&= F\left( \frac{\partial^2}{\partial x^2} g_\sigma(x, y) \right) F(I) + F\left( \frac{\partial^2}{\partial y^2} g_\sigma(x, y) \right) F(I) \\
&= F\left[ \frac{\partial^2}{\partial x^2} g_\sigma(x, y) + \frac{\partial^2}{\partial y^2} g_\sigma(x, y) \right] F(I) \\
&= F[\Delta^2 g_\sigma(x, y)] F(I) \\
\Rightarrow F(L) &= F[(\Delta^2 g_\sigma(x, y)) * I] \\
\Rightarrow L &= (\Delta^2 g_\sigma(x, y)) * I
\]