Solution for Assignment #2

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Prove the following results

1 $g_{\sigma_1}(x) * (g_{\sigma_2}(x) * I) = g_\sigma(x) * I$**. Also, find the value of** σ **in terms of** σ_1 **and** σ_2**.**

Proof:

$$
\begin{aligned}
L &= g_{\sigma_1}(x) * (g_{\sigma_2}(x) * I) \\
\Rightarrow F(L) &= F\left(g_{\sigma_1}(x) * (g_{\sigma_2}(x) * I)\right) \\
&= F\left(g_{\sigma_1}(x)\right) F\left((g_{\sigma_2}(x) * I)\right) && \text{By Convolution Theorem} \\
&= F\left(g_{\sigma_1}(x)\right) F\left(g_{\sigma_2}(x)\right) F\left(I\right) && \text{By Convolution Theorem} \\
&= \tfrac{\sqrt{2\pi}}{\sigma_1} g_{\frac{1}{\sigma_1}}(u) \tfrac{\sqrt{2\pi}}{\sigma_2} g_{\frac{1}{\sigma_2}}(u) F\left(I\right) && \text{Since } F\left(g_\sigma(x)\right) = \tfrac{\sqrt{2\pi}}{\sigma} g_{\frac{1}{\sigma}}(x) \\
&= \left[\tfrac{\sqrt{2\pi}}{\sigma_1} \tfrac{\sigma_1}{\sqrt{2\pi}} \exp\left(-\tfrac{u^2\sigma_1^2}{2}\right)\right] \left[\tfrac{\sqrt{2\pi}}{\sigma_2} \tfrac{\sigma_2}{\sqrt{2\pi}} \exp\left(-\tfrac{u^2\sigma_2^2}{2}\right)\right] F\left(I\right) \\
&= \exp\left(-\tfrac{u^2\left(\sigma_1^2+\sigma_2^2\right)}{2}\right) F\left(I\right) \\
&= \exp\left(\tfrac{-u^2\sigma^2}{2}\right) F\left(I\right) && \text{where } \sigma = \sqrt{\sigma_1^2+\sigma_2^2} \\
&= \tfrac{\sqrt{2\pi}}{\sigma} \left[\tfrac{\sigma}{\sqrt{2\pi}} \exp\left(-\tfrac{u^2\sigma^2}{2}\right)\right] F\left(I\right) \\
&= \tfrac{\sqrt{2\pi}}{\sigma} g_{\frac{1}{\sigma}}(u) F\left(I\right) \\
&= F\left(g_\sigma(x)\right) F\left(I\right) \\
\Rightarrow F(L) &= F\left(g_\sigma(x) * I\right) \\
\Rightarrow L &= g_\sigma(x) * I
\end{aligned}
$$

2. $g_\sigma(x,y) = g_\sigma(x) g_\sigma(y)$**.**

Proof:

$$
\begin{aligned}
g_\sigma(x,y) &= \tfrac{1}{2\pi\sigma^2} \exp\left(-\tfrac{x^2+y^2}{2\sigma^2}\right) \\
&= \left(\tfrac{1}{\sqrt{2\pi}\sigma}\right) \left(\tfrac{1}{\sqrt{2\pi}\sigma}\right) \exp\left(-\tfrac{x^2}{2\sigma^2}\right) \exp\left(-\tfrac{y^2}{2\sigma^2}\right) \\
&= \left[\left(\tfrac{1}{\sqrt{2\pi}\sigma}\right) \exp\left(-\tfrac{x^2}{2\sigma^2}\right)\right] \left[\left(\tfrac{1}{\sqrt{2\pi}\sigma}\right) \exp\left(-\tfrac{y^2}{2\sigma^2}\right)\right] \\
&= g_\sigma(x) g_\sigma(y)
\end{aligned}
$$

3. $g_\sigma(x,y) * I = g_\sigma(x) * (g_\sigma(y) * I)$**.**

Proof:

Method 1.

$$
\begin{aligned}
L &= g_\sigma(x,y) * I \\
\Rightarrow F(L) &= F\left(g_\sigma(x,y) * I\right) \\
&= F\left(g_\sigma(x,y)\right) F\left(I\right) && \text{By Convolution Theorem} \\
&= \tfrac{2\pi}{\sigma^2} g_{\frac{1}{\sigma}}(u,v) F\left(I\right) \\
&= \left(\tfrac{\sqrt{2\pi}}{\sigma}\right)\left(\tfrac{\sqrt{2\pi}}{\sigma}\right) g_{\frac{1}{\sigma}}(u) g_{\frac{1}{\sigma}}(v) F\left(I\right) && \text{By Problem 2} \\
&= \left(\tfrac{\sqrt{2\pi}}{\sigma} g_{\frac{1}{\sigma}}(u)\right)\left(\tfrac{\sqrt{2\pi}}{\sigma} g_{\frac{1}{\sigma}}(v)\right) F\left(I\right) \\
&= F\left(g_\sigma(x)\right) F\left(g_\sigma(y)\right) F\left(I\right) \\
&= F\left(g_\sigma(x)\right) F\left(g_\sigma(y) * I\right) \\
\Rightarrow F(L) &= F\left(g_\sigma(x) * (g_\sigma(y) * I)\right) \\
\Rightarrow L &= g_\sigma(x) * (g_\sigma(y) * I)
\end{aligned}
$$

Method 2.

$$
\begin{aligned}
g_\sigma(x,y) * I &= \int\int g_\sigma\left(x-x', y-y'\right) I\left(x',y'\right) dy' dx' \\
&= \int\int g_\sigma\left(x-x'\right) g_\sigma\left(y-y'\right) I\left(x',y'\right) dy' dx' && \text{By Problem 2} \\
&= \int g_\sigma\left(x-x'\right) \left[\int g_\sigma\left(y-y'\right) I\left(x',y'\right) dy'\right] dx' \\
&= \int g_\sigma\left(x-x'\right) \left(g_\sigma(y) * I\right)\left(x',y'\right) dx' \\
&= g_\sigma\left(x\right) * \left(g_\sigma(y) * I\right)
\end{aligned}
$$

4. $\Delta\left(g_\sigma(x,y) * I\right) = \left(\Delta g_\sigma(x,y)\right) * I$**.**

Proof:

$$
\begin{aligned}
L &= \tfrac{\partial}{\partial x}\left(g_\sigma(x,y) * I\right) \\
\Rightarrow F(L) &= F\left(\tfrac{\partial}{\partial x}\left(g_\sigma(x,y) * I\right)\right) \\
&= uF\left(g_\sigma(x,y) * I\right) && \text{Since } F\left(\tfrac{\partial}{\partial x} f\right) = uF(f) \\
&= uF\left(g_\sigma(x,y)\right) F\left(I\right) && \text{By Convolution Theorem} \\
&= \left[uF\left(g_\sigma(x,y)\right)\right] F\left(I\right) \\
&= F\left(\tfrac{\partial}{\partial x} g_\sigma(x,y)\right) F\left(I\right) \\
\Rightarrow F(L) &= F\left[\left(\tfrac{\partial}{\partial x} g_\sigma(x,y)\right) * I\right] \\
\Rightarrow L &= \left(\tfrac{\partial}{\partial x} g_\sigma(x,y)\right) * I
\end{aligned}
$$

Similarly, we can prove that $\frac{\partial}{\partial y}\left(g_\sigma(x,y) * I\right) = \left(\frac{\partial}{\partial y} g_\sigma(x,y)\right) * I$.

Hence, $\Delta\left(g_\sigma(x,y) * I\right) = \left[\frac{\partial}{\partial x}\left(g_\sigma(x,y) * I\right) \quad \frac{\partial}{\partial y}\left(g_\sigma(x,y) * I\right)\right]^T = \left[\left(\frac{\partial}{\partial y} g_\sigma(x,y)\right) * I \quad \left(\frac{\partial}{\partial y} g_\sigma(x,y)\right) * I\right]^T = \left[\left(\frac{\partial}{\partial y} g_\sigma(x,y)\right) \quad \left(\frac{\partial}{\partial y} g_\sigma(x,y)\right)\right]^T * I = \left(\Delta g_\sigma(x,y)\right) * I$

5. $|\Delta|$ is isotropic (rotation invariant).

Proof: Let $I(x, y)$ be an image and $I(x', y')$ be the rotated image such that $[x'\ y']^T = R[x\ y]^T$ where R is a 2D rotation matrix. Since $R^{-1} = R^T$, we have, $[x\ y]^T = R^T[x'\ y']^T$. Hence,

$$x = x' \cos\theta + y' \sin\theta \qquad y = -x' \sin\theta + y' \cos\theta$$

Now the derivatives of I in rotated coordinate system are given by

$$\frac{\partial I}{\partial x'} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial I}{\partial x}\cos\theta - \frac{\partial I}{\partial y}\sin\theta$$

$$\frac{\partial I}{\partial y'} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial y'} = \frac{\partial I}{\partial x}\sin\theta + \frac{\partial I}{\partial y}\cos\theta$$

The magnitude of gradient $|\Delta|$ in rotated coordinate system is given by

$$\begin{aligned}
|\Delta|(x', y') &= \sqrt{\left(\tfrac{\partial I}{\partial x'}\right)^2 + \left(\tfrac{\partial I}{\partial y'}\right)^2} \\
&= \sqrt{\left(\tfrac{\partial I}{\partial x}\cos\theta - \tfrac{\partial I}{\partial y}\sin\theta\right)^2 + \left(\tfrac{\partial I}{\partial x}\sin\theta + \tfrac{\partial I}{\partial y}\cos\theta\right)^2} \\
&= \sqrt{\left(\tfrac{\partial I}{\partial x}\cos\theta\right)^2 - 2\tfrac{\partial I}{\partial x}\tfrac{\partial I}{\partial y}\cos\theta\sin\theta + \left(\tfrac{\partial I}{\partial y}\sin\theta\right)^2 + \left(\tfrac{\partial I}{\partial x}\sin\theta\right)^2 + 2\tfrac{\partial I}{\partial x}\tfrac{\partial I}{\partial y}\cos\theta\sin\theta + \left(\tfrac{\partial I}{\partial y}\cos\theta\right)^2} \\
&= \sqrt{\left(\tfrac{\partial I}{\partial x}\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(\tfrac{\partial I}{\partial y}\right)^2\left(\cos^2\theta + \sin^2\theta\right)} \\
&= \sqrt{\left(\tfrac{\partial I}{\partial x}\right)^2 + \left(\tfrac{\partial I}{\partial y}\right)^2} \\
&= |\Delta|(x, y)
\end{aligned}$$

6. Δ^2 is linear and isotropic (rotation invariant).

Proof:

Linearity Let $f(x, y)$ and $g(x, y)$ be two functions and k be any scalar, then we have,

$$\begin{aligned}
\Delta^2(f + g) &= \tfrac{\partial^2}{\partial x^2}(f + g) + \tfrac{\partial^2}{\partial y^2}(f + g) \\
&= \tfrac{\partial^2}{\partial x^2}f + \tfrac{\partial^2}{\partial x^2}g + \tfrac{\partial^2}{\partial y^2}f + \tfrac{\partial^2}{\partial y^2}g \qquad \text{By linearity of derivative} \\
&= \left[\tfrac{\partial^2}{\partial x^2}f + \tfrac{\partial^2}{\partial y^2}f\right] + \left[\tfrac{\partial^2}{\partial x^2}g + \tfrac{\partial^2}{\partial y^2}g\right] \\
&= \Delta^2 f + \Delta^2 g
\end{aligned}$$

Also,

$$\begin{aligned}
\Delta^2(kf) &= \tfrac{\partial^2}{\partial x^2}(kf) + \tfrac{\partial^2}{\partial y^2}(kf) \\
&= k\tfrac{\partial^2}{\partial x^2}f + k\tfrac{\partial^2}{\partial y^2}f \qquad \text{By linearity of derivative} \\
&= k\left[\tfrac{\partial^2}{\partial x^2}f + \tfrac{\partial^2}{\partial y^2}f\right] \\
&= k\Delta^2 f
\end{aligned}$$

Hence, Δ^2 is linear.

Rotational Invariance

Let $I(x, y)$ be an image and $I(x', y')$ be the rotated image such that $[x'\ y']^T = R[x\ y]^T$ where R is a 2D rotation matrix. Since $R^{-1} = R^T$, we have, $[x\ y]^T = R^T[x'\ y']^T$. Hence,

$$x = x' \cos\theta + y' \sin\theta \qquad y = -x' \sin\theta + y' \cos\theta$$

Now the derivatives of I in rotated coordinate system are given by

$$I_{x'} = \frac{\partial I}{\partial x'} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial x'} = \frac{\partial I}{\partial x}\cos\theta - \frac{\partial I}{\partial y}\sin\theta$$

$$I_{y'} = \frac{\partial I}{\partial y'} = \frac{\partial I}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial I}{\partial y}\frac{\partial y}{\partial y'} = \frac{\partial I}{\partial x}\sin\theta + \frac{\partial I}{\partial y}\cos\theta$$

Similarly the second derivatives of I in rotated coordinate system are computed as follows:

$$\begin{aligned}
\frac{\partial I^2}{\partial x'^2} &= \frac{\partial I_{x'}}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial I_{x'}}{\partial y}\frac{\partial y}{\partial x'} \\
&= \frac{\partial}{\partial x}\left[\frac{\partial I}{\partial x}\cos\theta - \frac{\partial I}{\partial y}\sin\theta\right]\cos\theta - \frac{\partial}{\partial y}\left[\frac{\partial I}{\partial x}\cos\theta - \frac{\partial I}{\partial y}\sin\theta\right]\sin\theta \\
&= \left[\frac{\partial^2 I}{\partial x^2}\cos\theta - \frac{\partial^2 I}{\partial x \partial y}\sin\theta\right]\cos\theta - \left[\frac{\partial^2 I}{\partial x \partial y}\cos\theta - \frac{\partial^2 I}{\partial y^2}\sin\theta\right]\sin\theta \\
&= \frac{\partial^2 I}{\partial x^2}\cos^2\theta - \frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta - \frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\sin^2\theta \\
&= \frac{\partial^2 I}{\partial x^2}\cos^2\theta - 2\frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\sin^2\theta
\end{aligned}$$

$$\begin{aligned}
\frac{\partial I^2}{\partial y'^2} &= \frac{\partial I_{y'}}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial I_{y'}}{\partial y}\frac{\partial y}{\partial y'} \\
&= \frac{\partial}{\partial x}\left[\frac{\partial I}{\partial x}\sin\theta + \frac{\partial I}{\partial y}\cos\theta\right]\sin\theta + \frac{\partial}{\partial y}\left[\frac{\partial I}{\partial x}\sin\theta + \frac{\partial I}{\partial y}\cos\theta\right]\cos\theta \\
&= \left[\frac{\partial^2 I}{\partial x^2}\sin\theta + \frac{\partial^2 I}{\partial x \partial y}\cos\theta\right]\sin\theta + \left[\frac{\partial^2 I}{\partial x \partial y}\sin\theta + \frac{\partial^2 I}{\partial y^2}\cos\theta\right]\cos\theta \\
&= \frac{\partial^2 I}{\partial x^2}\sin^2\theta + \frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\cos^2\theta \\
&= \frac{\partial^2 I}{\partial x^2}\sin^2\theta + 2\frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\cos^2\theta
\end{aligned}$$

Finally, the Laplacian of the image in rotated coordinate system is given by

$$\begin{aligned}
\Delta^2 I(x', y') &= \frac{\partial I^2}{\partial x'^2} + \frac{\partial I^2}{\partial y'^2} \\
&= \left[\frac{\partial^2 I}{\partial x^2}\cos^2\theta - 2\frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\sin^2\theta\right] + \left[\frac{\partial^2 I}{\partial x^2}\sin^2\theta + 2\frac{\partial^2 I}{\partial x \partial y}\sin\theta\cos\theta + \frac{\partial^2 I}{\partial y^2}\cos^2\theta\right] \\
&= \frac{\partial^2 I}{\partial x^2}\left(\cos^2\theta + \sin^2\theta\right) + \frac{\partial^2 I}{\partial y^2}\left(\cos^2\theta + \sin^2\theta\right) \\
&= \frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} \\
&= \Delta^2 I(x, y)
\end{aligned}$$

7. $\Delta^2\left(g_\sigma(x,y)*I\right)=\left(\Delta^2 g_\sigma(x,y)\right)*I$**.**

Proof:

$$
\begin{aligned}
L &= \Delta^2\left(g_\sigma(x,y)*I\right)\\
\Rightarrow F(L) &= F\left(\Delta^2\left(g_\sigma(x,y)*I\right)\right)\\
&= F\left(\frac{\partial^2}{\partial x^2}\left(g_\sigma(x,y)*I\right)+\frac{\partial^2}{\partial y^2}\left(g_\sigma(x,y)*I\right)\right)\\
&= F\left(\frac{\partial^2}{\partial x^2}\left(g_\sigma(x,y)*I\right)\right)+F\left(\frac{\partial^2}{\partial y^2}\left(g_\sigma(x,y)*I\right)\right) && \text{By linearity of Fourier Transform}\\
&= u^2F\left(g_\sigma(x,y)*I\right)+v^2F\left(g_\sigma(x,y)*I\right) && \text{Since } F\left(\frac{\partial^n}{\partial x^n}f\right)=u^nF(f)\ \&\ F\left(\frac{\partial^n}{\partial y^n}f\right)=v^nF(f)\\
&= u^2F\left(g_\sigma(x,y)\right)F\left(I\right)+v^2F\left(g_\sigma(x,y)\right)F\left(I\right) && \text{By Convolution Theorem}\\
&= \left[u^2F\left(g_\sigma(x,y)\right)\right]F\left(I\right)+\left[v^2F\left(g_\sigma(x,y)\right)\right]F\left(I\right)\\
&= F\left(\frac{\partial^2}{\partial x^2}g_\sigma(x,y)\right)F\left(I\right)+F\left(\frac{\partial^2}{\partial y^2}g_\sigma(x,y)\right)F\left(I\right)\\
&= \left[F\left(\frac{\partial^2}{\partial x^2}g_\sigma(x,y)\right)+F\left(\frac{\partial^2}{\partial y^2}g_\sigma(x,y)\right)\right]F\left(I\right)\\
&= F\left[\frac{\partial^2}{\partial x^2}g_\sigma(x,y)+\frac{\partial^2}{\partial y^2}g_\sigma(x,y)\right]F\left(I\right)\\
&= F\left[\Delta^2 g_\sigma(x,y)\right]F\left(I\right)\\
\Rightarrow F(L) &= F\left[\left(\Delta^2 g_\sigma(x,y)\right)*I\right]\\
\Rightarrow L &= \left(\Delta^2 g_\sigma(x,y)\right)*I
\end{aligned}
$$