Let

$I$ be an image,

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad g_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right).$$

$$\Delta I = \begin{bmatrix} \frac{\partial}{\partial x} I & \frac{\partial}{\partial y} I \end{bmatrix}^T$$

is the Gradient of image $I$ and $\Delta^2 I = \frac{\partial^2}{\partial x^2} I + \frac{\partial^2}{\partial y^2} I$ is the Laplacian of image $I$.

$f * g$ denotes the convolution of $f$ with $g$.

1. Proof the following results

   1. $g_{\sigma_1}(x) * (g_{\sigma_2}(x) * I) = g_{\sigma}(x) * I$. Also, find the value of $\sigma$ in terms of $\sigma_1$ and $\sigma_2$.

   2. $g_\sigma(x, y) = g_\sigma(x)g_\sigma(y)$.

   3. $g_\sigma(x, y) * I = g_\sigma(x) * (g_\sigma(y) * I)$.

   4. $\Delta g_\sigma(x, y) * I = (\Delta g_\sigma(x, y)) * I$.

   5. $|\Delta|$ is isotropic (rotation invariant).

   6. $\Delta^2$ is linear and isotropic (rotation invariant).

   7. $\Delta^2(g_\sigma(x, y) * I) = (\Delta^2 g_\sigma(x, y)) * I$.

You may want to use the following results or facts.

1D Convolution: $f * g = \int f(x - x')g(x')dx'$

2D Convolution: $f * g = \iint f(x - x', y - y')g(x', y')dx'dy'$

Convolution Theorem: $F(f * g) = F(f)F(g)$ and $F(fg) = F(f)*F(g)$, where $F(f)$ is Fourier transform of function $f$.

Fourier Transform of Gaussian: $F(g_\sigma(x)) = \frac{\sqrt{2\pi}}{\sigma} g_{1/\sigma}(u)$, $F(g_\sigma(x, y)) = \frac{2\pi}{\sigma^2} g_{1/\sigma}(u, v)$