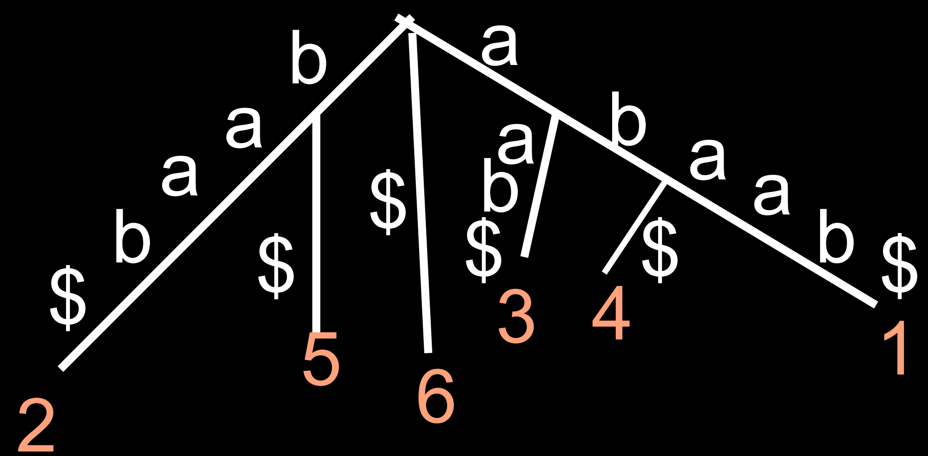
Suffix Tree

A suffix tree is a compressed trie of all suffixes of a string x$
x=abaab (x has length *m*-1, $ is an unique sentinel character, thus x$ has a length *m)*.

1: abaab$
2: baab$
3: aab$
4: ab$
5: b$
6: $



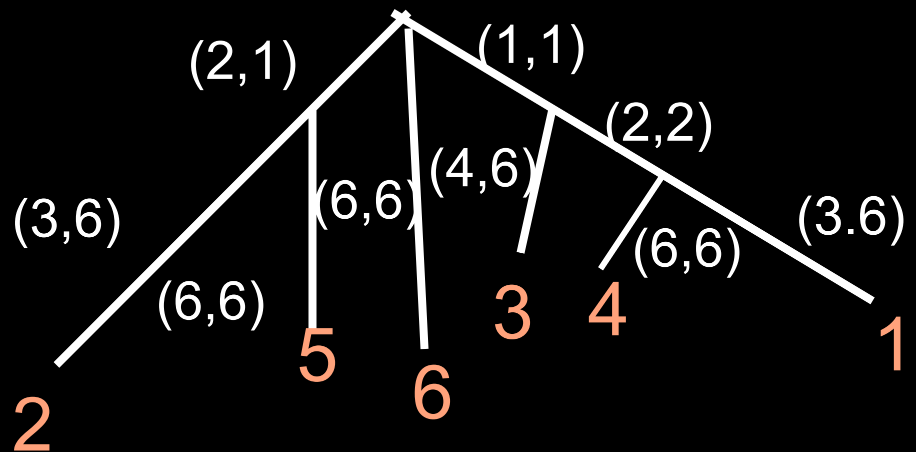
Each leaf node is marked with an integer j corresponding to suffix S[j…m]

$O(m)$ Storage Suffix Tree

A suffix tree is a compressed trie of all suffixes of a string x$

The path labels are specified by two integers (k,l), k=start index or position of the path and l= end position of the path.

1: abaab$
2: baab$
3: aab$
4: ab$
5: b$
6: $

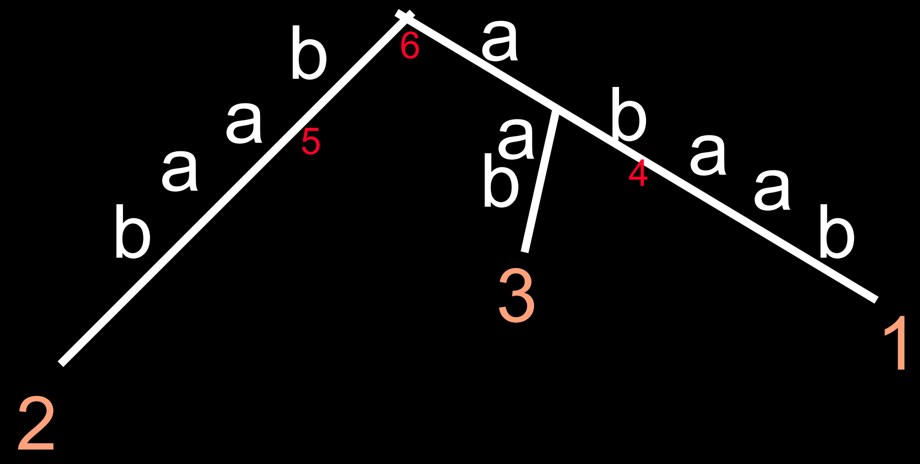


Each leaf node is marked with an integer j corresponding to suffix S[j…m]

Implicit Suffix Tree

- Remove all $. Remove all edges without any label. Remove all nodes with a single child and the merge the path labels into one label.

1: abaab$
2: baab$
3: aab$
4: ab$
5: b$
6: $



Suffixes 4,5 and 6 appear implicitly.

Ukkonen's Algorithm

- This an on-line algorithm. Given the sequence S[1,2,….m], it constructs implicit suffix trees I_i for the prefix S[1,2,…i] starting from I_1 incrementing i by 1until I_m is built. There are m phases. In phase i+1 I_{i+1} is constructed from I_i . The (i+1) phase has (i+1) extensions. Extension j (1<=j<=i+1) deal with the sequence S[j,…(i+1)]

High-Level Description

- Construct I_1
- For i from 1 to m-1

(begin phase i+1)

For j from 1 to i+1

(begin extension j)

Walk down the tree from root along the path S[j…i] in the current implicit tree. If needed, add S[i+1] at the end of the path to insure S[j…i+1] is in the tree.

end

end

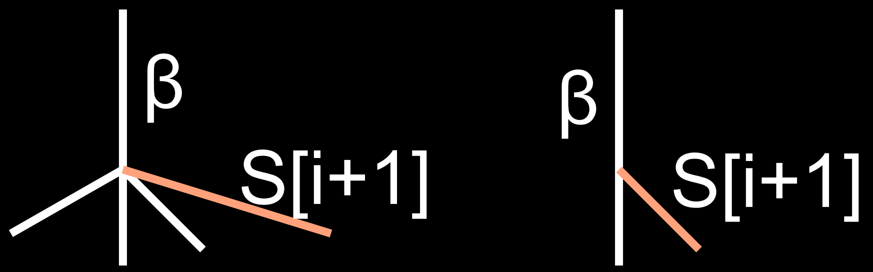
Types of Nodes

- Leaf Node: corresponds to a suffix
- Explicit Node: Internal node that has at least two branches
- Implicit Nodes: corresponds to a suffix but due to path compression resulting from elimination of $, it only has one child and hence has been reduced to an implicit node.

We will use Greek symbols α, β, γ to denote strings and symbols x,y,z to denote single characters. Let β=S[j…i] be a suffix of S[1…i]

Suffix Extension Rules

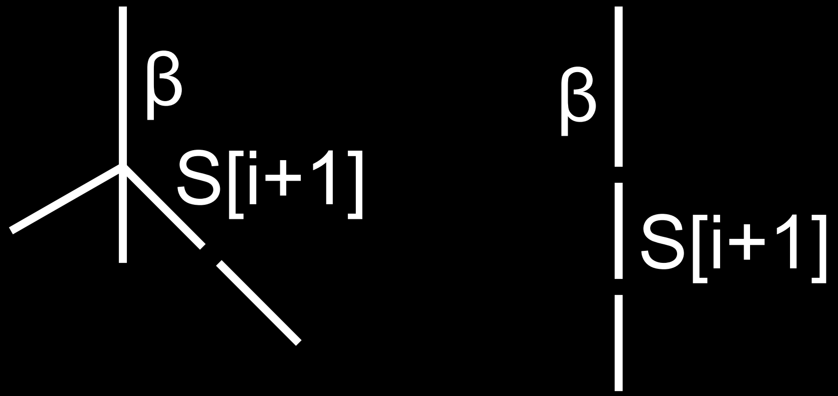
- Rule 1: Once a leaf, always a leaf.
- Rule 2: No path from end of string β in I_i starts with S[i+1] but the path continues. Then attach an edge with label S[i+1] at the end of β creating an explicit node if necessary.



(The 'white' tree is in I_i. White plus purple is in I_{i+1})

Extension Rule 3

- Rule 3: Some path from the end of β starts with S[i+1] so βS[i+1] is already in the tree. Do nothing.

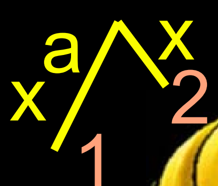


An example: S=axabxb

Phase i=1 (Note, in phase i=0, the tree simply a single node, the root corresponding to the empty string ε)

β=ε; Only one extension. βS[i+1]=a. Rule 2 is applicable and I_1 is

Phase i=2 Two extensions for suffixes ax and x . Suffix ax is inserted using Rule 1 and x in inserted by using Rule 2



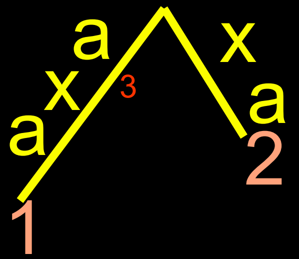
(Rule Sequence:12)

Example (continued)

Phase i=3

Three extensions for suffixes axa, xa and a. Note, the first two extensions are simply copying I_2 and adding last character S[i+1]=a to it at the leaf nodes by applying Rule 1. Then the last suffix is a single character a which is handled, in this case, by Rule 3 (do nothing).

(Rule Sequence: 113)

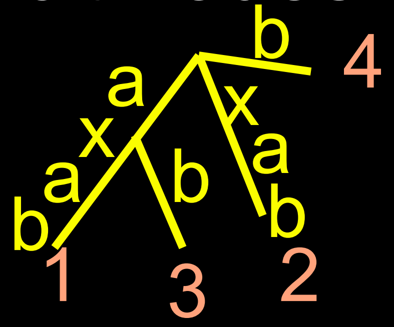


Phase 4

Phase i=4

The four suffixes are axab,xab,ab and b

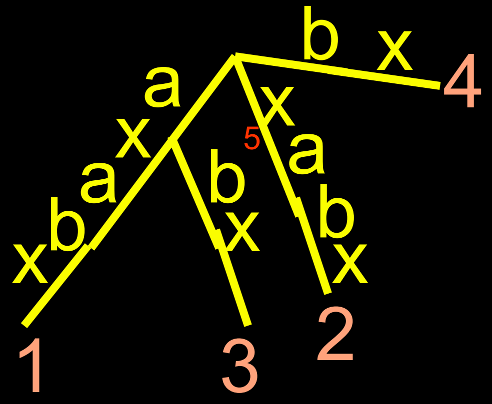
The Rule sequence (1122). Note due to Rule 2, two explicit nodes have been added.



Phase 5

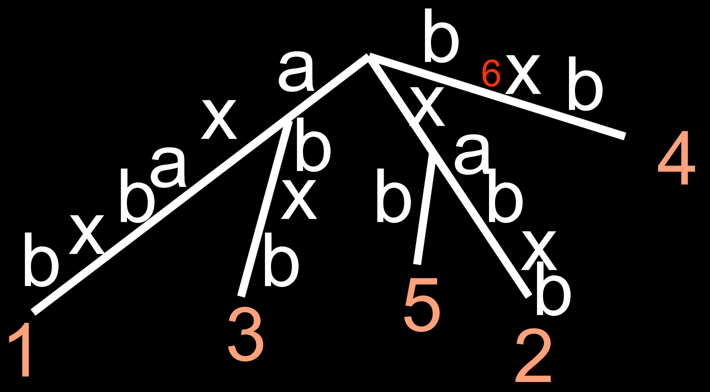
- Phase i=5
- The sequencesare axabx, xabx,abx.bx and x.
- Rule Sequence (11113)

(Node 5 isimplicit)



Phase 6

- Phase i=6
- The suffixes are axabxb,xabxb,abxb, bxb,xb and b



The Rule

Sequence: (111123)

Properties of Rule Sequence

- A very important property of the Rule sequence is that it consists of an initial sequence of 1's (except for phase 1 which has one Rule 2) followed by possibly a sequence of 2's and if a 3 occurs at the end, further application of Rule 3 can be abandoned.
- Another property is that the length of the Rule 1 sequence in phase i+1 is equal or strictly one more than the length of Rule 1 sequence in phase i.

(Justify the above statements)

The Naïve Algorithm

- In the naïve algorithm that follows from the 'High Level Description', once we locate the end of the string β in the current tree I_i , inserting S[i+1] after it, takes constant amount of work. The crux of the problem is to find out where β ends in the current tree. For this, we engaged in walking down the tree matching characters taking O(|β|) time for |β|=i,i-1,….,2,1 for the (i+1) phase.
- So, I_{i+1} is created from I_i making $O(i^2)$ character comparisons. So, total number of comparisons for all phases is

123   i i+1

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-

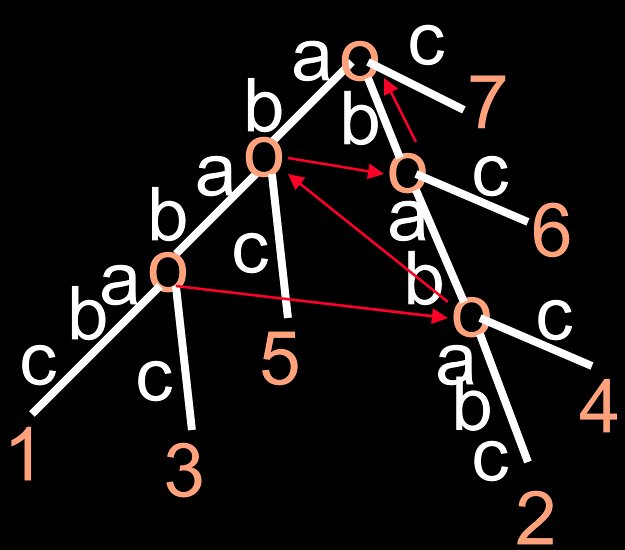
$$\sum_{i=1}^{m} o(i^2) = O(m^3)$$

Suffix Link- how to be lazy but smart (or walking is good for your health but not for your algorithm)

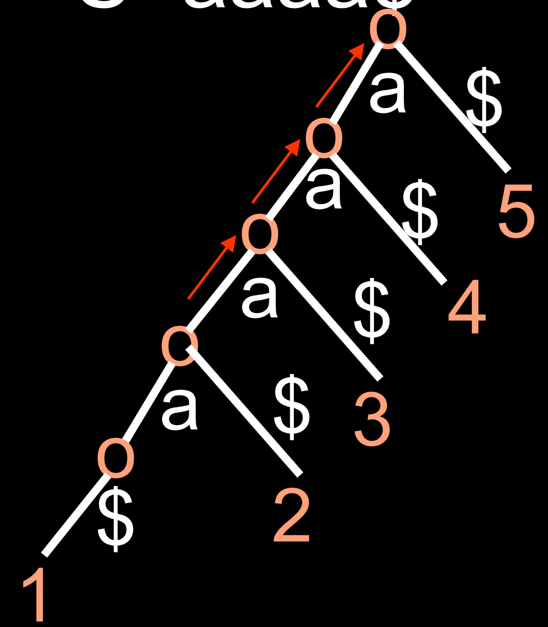
- To avoid walking down the tree for each β, suffix links are introduced
- Definition: Let an internal node v has a path label x α where x is a character and α is a a string (possibly empty). If there is another node s(v) in the tree with path label α, then a pointer from v to s(v) is called a suffix link, denoted (v,s(v)). If α is empty string ε, the suffix with path label x goes to the root. The root is not considered internal and has no suffix link.

Suffix Link Examples

S=abababc



S=aaaa$



Suffix Link Creation

- Lemma: Let an internal node v with path label x α be added to the current tree I_i in extension j of the (i+1) phase. Then
- Either an internal node with edgelabel α already exits in the current tree I_i
- Or an internal node with path label αwill be created in extension j+1 in the same phase i+1.

Suffix Link Creation

- Corollary 1: In Ukkonen's algorithm, any newly created internal node will have a suffix link from it by the end of the next extension.
- Corollary 2: In any suffix tree I_i , if an internal node v has a path label x α, then there is a node s(v) of I_i with path label α.

First Extension

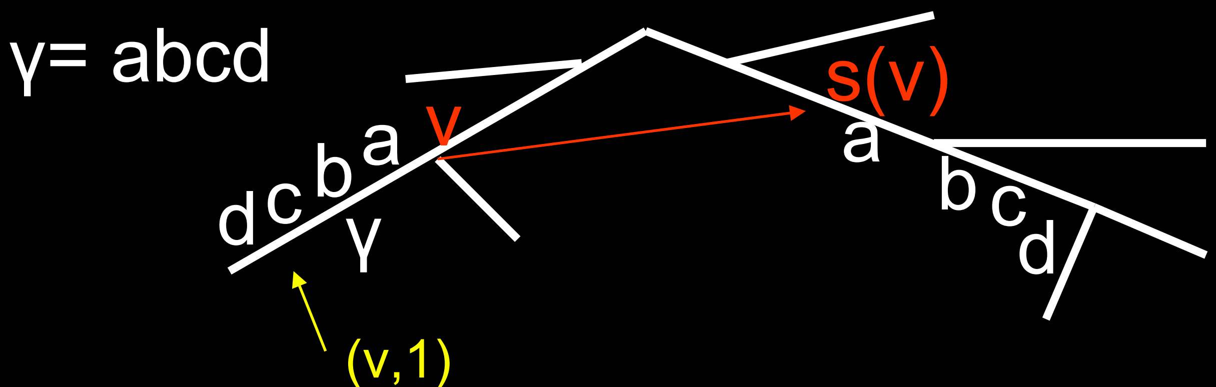
- Using Rule 1 (once a leaf, always a leaf), the first extension can be done in constant time. Keep a pointer to the leaf node 1 of current tree I_i corresponding to S[1…i]. Just add s[i+1] at the end , node label still remains 1 and the pointer is adjusted to point to new node 1.

Second Extension

- Let S[1..i] =x α (X a character, α could be ε). To do the extension, we need to find the end of S[2..i]= α in the current tree I_i . Let (v,1) be the edge that enters leaf 1 { Node v could be the root [viz. for S=aaaa$] or an internal node.
- If v is the root, walk down the tree following the path label α.

Second Extension

- If v is an internal node, walk up from leaf 1 via edge (v,1) to node v.
- Follow the suffix link (v,s(v)).
- Walk from s(v) down the path checking for all the characters in the string γ which is the path label of (v,1). This journey may use more than one edge.
- Update the tree following the applicable extension rule at the end of the path (it could be Rule 1,2 or 3).



General Extension j>2

- The procedure is essentially the same as for j=2 except we start from string S[j-1..i] in the curent tree I_i and walk up at most one node to either root or node v, follow path γ to the end of S[j..i] and then extend the suffix to s[j..i+1] using the applicable extension rule.
- There is one difference: the end of S[j-1..i] may itself have a suffix link; then do not walk up any node, just follow the suffix link.
- If a new internal node was created in extension j-1(by extension Rule 2) then a string α must end at node s(w) [by Lemma]. Then create the suffix link (w, s(w)).

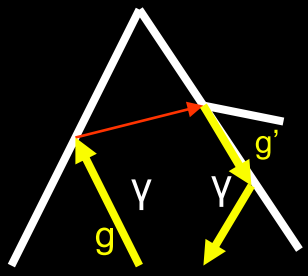
Single Extension Algorithm

- Find the first node v at or above the end of S[j-i..i] that either has a suffix link from it or is the root. This requires walking up at most one edge with label γ .
- If v is not the root , traverse suffix link (v,s(v)) and then walk down γ from s(v). If v is the root follow the path S[j..i] from root, as in the naïve algorithm.
- Using the extension rules, ensure that S[j..i]S[i+1] is in the tree.
- If a new internal node was created in extension j-1(by extension Rule 2) then a string α must end at node s(w) [by Lemma]. Then create the suffix link (w, s(w)).

The suffix link improves the performance in practice but so far the worst case time complexity is still $O(m^3)$.

Trick 1: Skip and Count

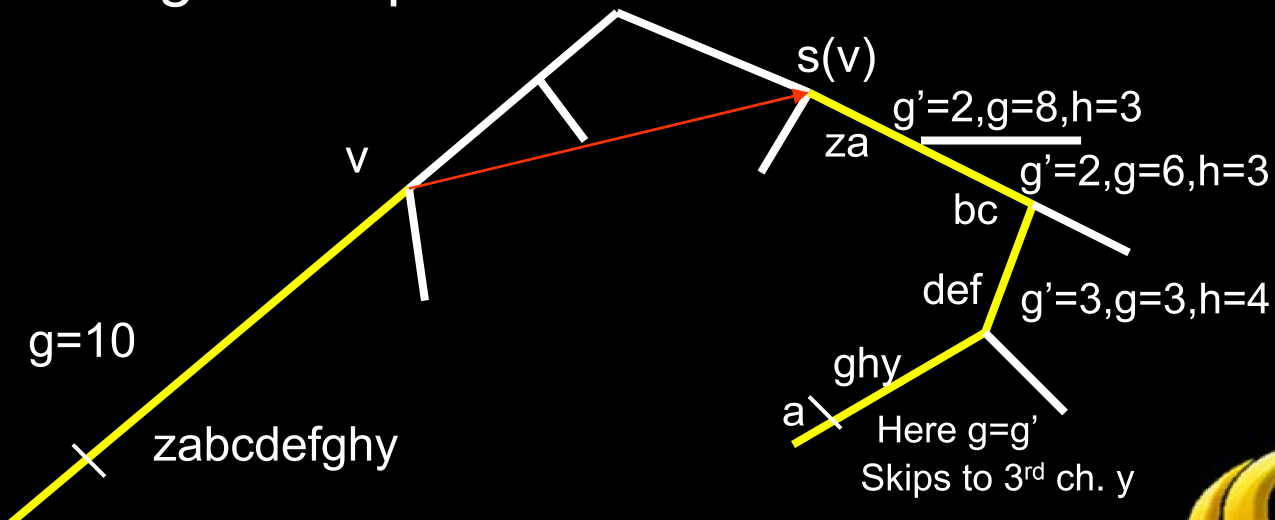
- The complexity of walking down γ from s(v) is O(|γ|). g=| γ|=number of character in γ. No two edges out of s(v) have the sme character; so the first charcater of γ appears in a unique path from s(v). Let g’ be the number of characters in the edge with this unique character. If g’<g the algorithm can



simply skip to the node at the end of the edge and set g<- g-g’
h<- g’+1
and look for h th character of γ for the next match to follow the downward path.

Skip and Count

- This process can be iterated for the succeeding edges as long as g'<g. When an edge is reached such that g<=g', the algorithm skips to character number g in the path.



Complexity of Skip and Count

- It should be obvious that moving from node to node using (k,l) labels and the g and h values on the γ path takes constant time. The total time to traverse the path is thus depends on the number of nodes traversed rather than the number of characters.
- Node Depth: Node depth of a node u is the number of nodes on the path from the root to u.
- Lemma: Let (v,s(v)) be the suffix link traversed at any time in the algorithm. At this time, the node depth of v is at most 1 greater than node depth of s(v)
- Theorem: Using skip and count technique, any phase of Ukkonen's algorithmtakes O(m) time
- Corollary: Ukkonen's algorithm can be implemented with suffix link to run in $O(m^2)$ time.

Trick 2: Rule 3 is a show stopper

- Rule 3 means do nothing because the path labeled s[j..i] continues with character S[i+1]. So, do the paths labeled S[j+1..i], S[j+2..i] ,..,S[i]. Thus if extension Rule 3 applies in extension j, the same Rule 3 must apply to all succeeding extensions in (i+1) phase. This leads to:
- Trick 2: End any phase i+1 the first time Rule 3 applies. If this happens in extendsion j, then there isno need to explicitly find the end of any string S[k..i] for k>j.

Extensions 1 in bulk

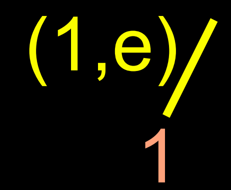
- First, to conserve storage, we are not going to write the character sequences in any edge. Instead we use a pair of indices (k,l) to limit storage to O(m).
- Second, due to "once a leaf, always a leaf" rule, once there is a leaf labeled j, extension Rule 1 will apply always to extension j in any successive phase.
- In any phase I, there is an initial sequence of consecutive extensions (starting with extension 1) where extension Rule 1 or 2 applies. Let j_i denote the last extension in phase i. It follows from "Once a leaf, always a leaf" that $j_i <= j_{i+1}$ That is, the initial sequence of extension rules 1 or 2 cannot shrink in successive phases.
- Let us take an example.

An example: S=axabxb

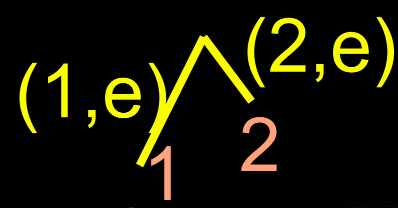
Phase i=1 (Note, in phase i=0, the tree simply a single node, the root corresponding to the empty string ε)

$\beta=\varepsilon$; Only one extension. $\beta S[i+1]=a$.

l_1 is with value of e=1



Phase i=2 Two extensions for suffixes ax and x . Suffix ax is inserted using Rule 1 and x in inserted by using also Rule 1. Root node is not considered an Internal node. (Rule Sequence:11)

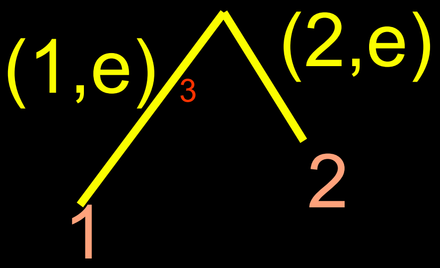


e=2

Example (continued)

Phase i=3

Three extensions for suffixes axa, xa and a. Note, the first two extensions are simply copying I_2 and adding last character S[i+1]=a to it at the leaf nodes by applying Rule 1. Then the last suffix is a single character a which is handled, in this case, by Rule 3 (do nothing).



(Rule Sequence: 113)

e=3

Phase 4

Phase i=4

The four suffixes are axab,xab,ab and b

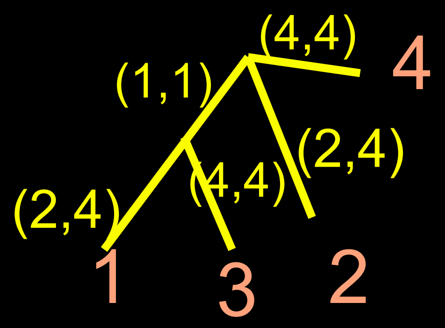
The Rule sequence is (1122). Note due to Rule 2, two explicit nodes have been

added. Thek values in (k.e)

for the edges are determined

at the time Rule 2 is applied.

Note e=4 for all leaf nodes.

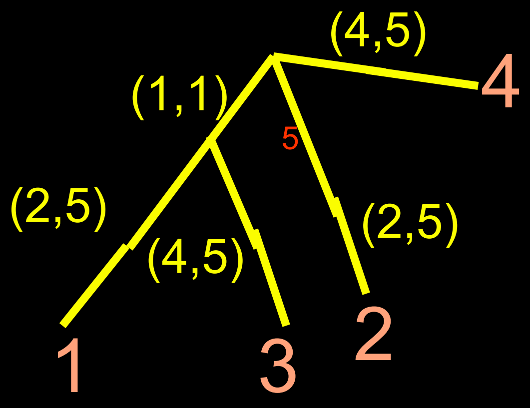


Phase 5

- Phase i=5
- The sequencesare axabx, xabx,abx.bx and x.
- Rule Sequence

  (11113) e=5

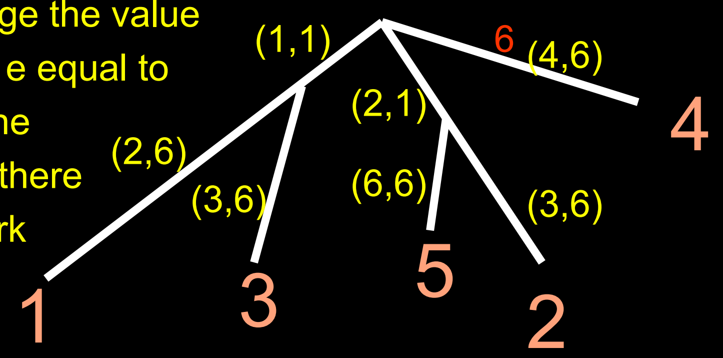
  (Node 5 is implicit)



Phase 6

- Phase i=6
- (Wait, wait!! You have been misleading us saying all these e-values are changed in O(m) times. If we change the value of e at every phase, it is going to take $O(m^2)$ time.)
- Here's the punch line! Don't change the value of e until the last phase and make e equal to the value of maximum phase for the i+1 extension, which is i+1. Since there are total of m extensions , the work involved in all the Rule 1 applications is

O(m)



The Rule

Sequence: (111123)

The suffixes are axabxb,xabxb,abxb, bxb,xb and b

Extension Sequences

The extension sequences for different phases are (1),(11),(113),(1122),(11113) and (111123). Note $j_1<j2=j3<j_4=j_5<j_{6..}$This suggests an implementation trick that avoids in phase i+1 all explicit extensions 1 through j_i. Only constant time will be needed to do all these extensions.

Trick Number 3

- In phase i+1, when a leaf is first created and would normally be labeled as S[p…i+1] written as (p,i+1) on the edge, write this as (p,e), where e (denoting "the current end") is a global variable to that is set to value i+1 once in each phase.
- In phase i+1, the algorithm knows that Rule 1 will apply in extensions 1 through j_i at least, only constant amount of work is needed up to extensions j_i to increment the value of e. The algorithm can then proceed to extension j_i+1 and perform explicit work if needed.

The Punch Line

- With tricks 2 and 3, explicit extensions in phase i+1 (using SEA – Single Extension algorithm) are only required from extension j_i+1 until the first Rule 3 applies or until i+1 is done. All other extensions before or after those explicit extensions, are done implicitly.

The Single phase Algorithm SPA

begin

1. Increment index e by i+1. By Trick 3, this correctly implements all implicit extensions 1 through $j_{i.}$

2. Explicitly compute successive extensions (using SEA) starting at j_i+1 until reaching first extension j^* where Rule 3 applies or until all extensions are done in this phase. By Trick 2 (show stopper), this correctly implements all the additional implicit extensions j^* +1 through j+1.

3. Set j_{i+1} to j^* to prepare for the next phase.

end