- **Two Types of Signals**
  -Continuous-time signal: $x(t)$, for any value of t, positive or negative.
  -Discrete-time signal: $x[n] = x(nT), n = 0, 1, 2, ., -1, -2, .,$ where T is called *sampling period.* The quantity $f_s = 1/T$ is called the sampling frequency. To faithfully reconstruct $x(t)$ from $x(n)$, we must have f_s greater than or equal to twice the maximum frequency in the signal(Shannon's theorem). Sampling a signal at a frequency less than f_s might introduce spurious *aliased signals.* To acomodate with a given maximum sampling frequency, it is common practice to intoduce a *low-pass* filter to purge the signal of all frequencies greater than half the maximum sampling frequeny. If for $t < 0$ or $n < 0$, the signal value is zero, then the signal is called a *causal signal.*

- **Continuous/Discrete Signal Processing System**
  -is a black box with *input* signal x and output *response* y. Designated as $x \to y$
  For a discrete system, if the values of both x and y are *quantized*, the system is called a *Digital Signal Processor (DSP).*

- **Operations on Signals**

1. **Delay:** signal $x(t)$ delayed by τ, designated by $x(t-\tau)$. signal delayed by k sampling period is designated by $x[n-k]$.
2. **Add Constant**: $x(t)+c$ or $x[n]+c$ where c is a constant.
3. **Multiplication by a scalar**: $ax(t)$, $ax[n]$, where a is a scalar.
4. **Summation**: $x_1(t)+x_2(t)$ or $x_1[n]+x_2[n]$

- **Linear System**

1. Superposition: If $x_1 \to y_1$ and $x_2 \to y_2$ then $x_1 + x_2 \to y_1 + y_2$
2. Homogeneity: If $x_1 \to y_1$, then $bx_1 \to by_1$ where b is a constant, real, imaginary or complex.

- **Time-invariance**
  If If $x(t) \to y(t)$ then $x(t-\tau) \to y(t-\tau))$.
  If If $x[n] \to y[n]$ then $x[n-k] \to y[n-k])$.

- **Recurrence Equation**
  Using the basic operations defined above, we can define a signal processing system whose output depends on present and past input/output as follows:

$$y[n] = \sum_{i=0}^{N} a_i x[n-i] - \sum_{k=1}^{M} b_k y[n-k]$$

This is called a *linear difference equation* or *recurrence equation*. The output is a funtion of past N inputs and M outputs. The effect is to reduce the rapidly varying components in the input signal and thus acting as a *low-pass filter*. If the second summation, the so-called *feedback* term ia absent, the system is called *non-recursive*. Otherwise, the system is called a *recursive* system.

- **Unit Sample Sequence or Delta Sequence,** $\delta[n]$

$$\delta[n] = 1, 0, 0, 0, 0, ...$$

Its value is 1 only when $n = 0$. For any other value of n, it is 0. A delayed delta function is 1 at sample delay k; everwhere else its value is 0. Thus,

$$\delta[n-3] = 0, 0, 1, 0, 0, 0, ...$$

Since multiplying by 1 does not change the value, the valuse of a signal at sample time k can be written as $x[k]\delta[n-k]$. Thus, we can write:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

And for continuous case the equation looks like:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

- **Convolution**

Let us say that we know the system response with respect to the unit-sample function as input. This is sufficient to compute the output resposene with respect to any arbitary input. Assume:

$$\delta[n] \rightarrow h[n]$$

Then by homogeneity property, we have:

$$a\delta[n] \rightarrow ah[n]$$

And by time-invariance property, we have:

$$a\delta[n-k] \rightarrow ah[n-k]$$

Now, by applying the principle of superposition, we have:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

We say that the resulting sequence $y[n]$ is the **convolution** of $x[n]$ with $h[n]$. By associativity, it can be verified that we get the same output response if we think of the input sequence as the response to unit-sample and substitute the unit-sample response by the input sequence. Thus, we can also write:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$$

The derivation of *convolution integral* for continuous signal is a bit more complicated because we have to introduce the equivalent of impulse response function called the Dirac Delta function. But the end result looks very similar:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau) = x(t) * h(t)$$
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) = h(t) * x(t)$$

- **Frequency Domain Model**

A time-varying sinusoidal signal like $x(t) = A\cos w_0 t$ can be written as

$$x(t) = \frac{A}{2}e^{jw_0 t} + \frac{A}{2}e^{-jw_0 t}$$

which gives a *frequency domain model* of $x(t)$ as consisting of two components of *amplitude* $\frac{A}{2}$ at the frequencies w_0 and $-w_0$ (never mind negative frequency - it is a model). If the signal had an initial *phase* angle of θ, that is, if $x(t) = A\cos(w_0 + \theta)$, then it can be expressed as

$$x(t) = [\frac{A}{2}e^{j\theta}]e^{jw_0t} + [\frac{A}{2}e^{-j\theta}]e^{-jw_0t}$$

which shows two components as before but with phase shifts θ and $-\theta$, respectively.

- **Frequency Model of an Exponential Signal**
  Let us re-write the convolution integral for $y(t)$:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau) = h(t) * x(t)$$

Now assume $x(t) = e^{jwt}$. Then $x(t-\tau)$ is equal to $e^{jwt}e^{-jw\tau}$. This yields:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{jwt}e^{-jw\tau}d\tau$$

Taking the term independent of τ out , we have

$$y(t) = e^{jwt}\int_{-\infty}^{\infty} h(\tau)e^{-jw\tau}d\tau$$

If we set

$$H(jw) = \int_{-\infty}^{\infty} h(\tau)e^{-jw\tau}d\tau$$

we can re-write $y(t)$ as

$$y(t) = e^{jwt}H(jw)$$

$H(jw)$ is in general a complex number with a magnitude $|H(jw)|$ and a phase $\theta(w)$. If the original signal were $x(t) = Ae^{jwt}$ with an amplitude A, we can write $y(t)$ as

$$y(t) = A|H(jw)|e^{j[wt+\theta(w)]}$$

Thus, the effect of the linear system on the exponential input is to modify the amplitude by a factor $|H(jw)|$ and the phase by an angle $\theta(w)$

- **Frequency Model of a Sinusoidal Signal**

  If we let $x(t) = A\cos wt$, as we have seen we can write $x(t)$ as:

$$x(t) = \frac{A}{2}e^{jw_0t} + \frac{A}{2}e^{-jw_0t}$$

Hence,

$$\frac{A}{2}e^{jw_0t} \rightarrow A|H(jw)|e^{j[wt+\theta(w)]}$$

and

$$\frac{A}{2}e^{-jw_0t} \rightarrow A|H(jw)|e^{-j[wt+\theta(w)]}$$

So, the response to $x(t)$ expressed as the superposition of the responses for its components yilds

$$y(t) = A|H(jw)|\cos[wt + \theta(w)]$$

This shows that the response of a linear system to a sinusoidal input is itself a sinusoid at the same frequency as input. The amplitude and phase of the response is determined by the input and the value of the frequency response-response function of the system at the input frequency.

$$H(jw) = \int_{-\infty}^{\infty} h(\tau) e^{-jw\tau} d\tau$$

- **The Fourier Transform**

The equation that we have encountered eralier relating $h(t)$ with $H(jw)$ (substituting t for τ)viz.

$$H(jw) = \int_{-\infty}^{\infty} h(t) e^{-jwt} dt$$

is called the **Fourier transform** $H(jw)$ of the impulse response signal $h(t)$. If we substitute $h(t)$ by $x(t)$ in the above equation, we call it a **Fourier transform** of $x(t)$. Note, this is only a mathematical transformation and $x(t)$ may not be any response to any other signal.Thus, we write

$$X(jw) = \int_{-\infty}^{\infty} x(t) e^{-jwt} dt$$