The arithmetic coders used in JPEG, JPEG 2000 and JBIG are called QM-coder[1]. It handles only binary strings or input and it is designed for simplicity and speed. It uses approximation for multiplication operation, fixed-precision integer arithmetic with renormalization of the probability interval from time to time.

The main idea of the QM-coder is to classify the input bit as **M**ore **P**robable **S**ymbol (*MPS)* and ***L**ess **P**robable **S**ymbol* (*LPS)*. Before the next bit is input, the QM-coder uses a statistical model (using a context, typically a two-dimensional context of black and white pixel in an image) to predict which one of the bits (0 or 1) will be the *MPS*. If the predicted *MPS* bit does not match with the actual bit, then the QM-coder will classify this as *LPS*; otherwise, it will continue to be classified as *MPS*. The output of the coder is simply a compressed version of a stream of *MPS* or *LPS*, which are assigned probability values dynamically. The decoder has only the knowledge of whether the next predicted bit is *MPS* or *LPS*. It uses the same statistical model as that of the encoder to obtain the actual values of the bit. Recall the range update equations we used for arithmetic coding: Let L and H denote the current 'low' and 'high', respectively, and current 'range' $A=H-L$. Let the current incoming symbol be a_i, its probability be $p(a_i)$ and its cumulative low and high probability be $P(a_{i-1})$ and $P(a_i)$, respectively, then the new low and new high become

$$L := L + (H - L) * P(a_{i-1})$$
$$H := L + (H - L) * P(a_i)$$

Then the new range becomes (by subtracting new L from new H)

$$A := A\,[P(a_i) - P(a_{i-1})] = A * p(a_i)$$

Let us assign a probability Q to *LPS* and assign the lower interval to *MPS* with

[1] There is an error in the book [Data Compression: The Complete Reference, 2nd Edition]: Page 121, Table 2.64: The renormalized values of C are wrong. The correct table is:

Symbol	C	A	Renor. A	Renor. C
Initially	0	1		
S1 (LPS)	0+1-0.1=0.9	0.1	0.8	0.9*2^3= 7.2
s2 (MPS)	unchanged 7.2	0.8-0.1=0.7	1.4	7.2*2= 14.4
s3 (LPS)	14.4+1.4=0.1=15.7	0.1	0.8	15.7*2^3=125.6
s4 (MPS)	unchanged 125.6	0.8-0.1=0.7	1.4	125.6*2=251.2

Table 2.64 Renormalization Added to Table 2.61.

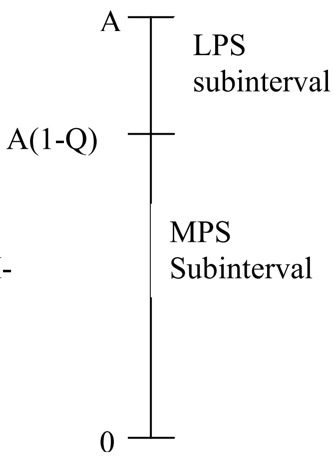
-----------From http://www.davidsalomon.name/DC2advertis/DC2errata.html

Symbol	Probability	Range
LPS	Q	[1-Q,1]
MPS	1-Q	[0,1-Q]

The value of Q is calculated dynamically using a statistical model. The output is simply a fractional value *C* indicating the state of the encoder, more specifically, the lower bound of the new sub-interval. If the next symbol is classified as MPS, the lower bound of the new interval remains unchanged, so C is unchanged. However, if the next symbol is LPS, the lower bound of the new sub-interval will be leveraged by the value of A(1-Q), so C is updated as C+A(1-Q).

The encoder executes the following code:

```
When MPS is encoded
        begin
            C is unchanged;
            A: = A (1 – Q)
        End
When LPS is encoded
        begin
            C: = C+A (1 – Q);
            A:= AQ
        End
```



Assignment of subintervals for QM-coder

Example:

Assume Q= 0.5. Input: (*LPS, MPS, LPS, MPS*). Initially, C=0 and A=1.

Step1: *LPS* : *C*=0+1(1-0.5)=0.5 *A*= 1*0.5= 0.5
Step2: *MPS*: *C*=0.5 *A*= 0.5(1-0.5) = 0.25
Step3: *LPS*: *C*=0.5+0.25(1-0.5) =0.625 *A*=0.25*0.5= 0.125
Step4: *MPS: C*=0.625 *A*=0.125(1-0.5) = 0.0625

Note, the range continues to shrink and the value of *C* gets bigger with longer input sequence.

The QM-decoder is just the reverse of the QM-encoder. We begin with initial value of *A* =1. The dividing line is *A*(1-*Q*)=1(1-0.5)=0.5. So, initially the (*MPS.LPS*) subintervals are [0,0.5) and [0.5,1), respectively.

Step1: C=0.625 falls in upper subinterval. So, an *LPS* is decoded. The new *C*=0.625-1(1-0.5)=0.125 and new *A*=1*0.5=0.5.

Step2: *C*=0.125, *A*=0.5. The dividing line is 0.5(1-0.5) =0.25. The (*MPS,LPS*) subintervals are [0, 0.25) and [0.25,0.5) , respectively. So, a *MPS* is decoded. The value of *C* remains unchanged and new *A*= 0.5(1-0.05)=0.25.

Step3: *C*=0.125, *A*=0.25. The dividing line is 0.25(1-0.5) =0.125. The (*MPS,LPS*) subintervals are [0, 0.125) and [0.125,0.25) , respectively. So, a *LPS* is decoded. The new *C*=0.125-0.25(1-0.5)=0 and new *A*=0.25*0.5=0.125.

Step4: *C*=0, *A*=0.125. The dividing line is 0.125(1-0.5) =0.0625. The (*MPS,LPS*) subintervals are [0, 0.0625) and [0.0625,0.125) , respectively. So, a *MPS* is decoded. The value of *C* remains unchanged and new *A*= 0.125(1-0.05)=0.0625.

The decoding algorithm can be written as

```
Input C, A=1 and Q;
Begin
      X:= A*(1-Q);
    MPS_interval=[0, X);
    LPS_interval= [X,A);
    If C is in MPS_interval then
      { Output MPS symbol;
          A:=A*(1- Q)}
     else
      { Output LPS symbol;
          C:=C- A*(1- Q);
          A:=A* Q)}
     endif;
```

end

If we take Q_e= 0.1, with the same input, the final value of C will be 0.981 and that of A 0.0081. If A gets too small, the precision of the machine will not be able to handle the fraction. The proposed remedy is to *renormalize* by doubling A and C (in effect, left shifting the binary integer by one bit representing the multiplication by 2). So, we need some threshold t such that when A is less than t, then A should be doubled. Furthermore, both A and 2*A should be close to 1.0. The mathematically representation for this requirement is:

Find t>0.5 such that we have min(max(1-t, 2t -1))

If t has the value of 0.9, then **max(1-t, 2t -1) = 0.8;** if t has the value of 0.6, then **max(1-t, 2t -1) = 0.4.** The solution to this problem is t = 0.75, and **max(1-t, 2t -1) = 0.25.**

The other problem is to compute the new value of the range A which needs multiplication. The JBIG committee recommended that the multiplication could be avoided by approximating the value of A to be close to 1, that is, $A\ (1 - Q_e)$ is approximated by $(A - Q_e)$ and AQ_e by Q_e. With the optimal threshold t = 0.75, the value of A is always maintained between [0.75, 1.5).

In order not to violate the assumption that A is close to 1, whenever A goes below 0.75 the QM coder goes through a series of rescaling until the value of A goes higher than 0.75. The rescaling operation is simply doubling which corresponds to a left shift if A is represented in binary. The same rescaling must be applied to C by doubling its value. Rescaling happens every time *LPS* occurs. For A, the rescaling occurs only if its value dips below 0.75. The modified encoder algorithm is:

```
When MPS is encoded
        begin
            C is unchanged;
            A: = A – Qe;
            If (A<0.75) then
            Renormalize A and C;
            Endif;
        End
When LPS is encoded
        begin
            C: = C+ (A - Qe);
            A:= Qe;
            Renormalize A and C;
        End
```

In the aforementioned pseudo-code, when LPS is encoded, A is updated as Qe, which is in practice always less than 0.5. So the condition of the re-normalization A < 0.75 holds. Therefore re-normalization is always conducted when we have a LPS symbol.

One problem introduced by approximated re-normalization is that the subinterval allocated to the MPS may be smaller than the sub-interval allocated to the LPS. This is because we use A= A-Qe to approximate A=A(1-Qe). For example, suppose we have Qe=0.45, and we have four MPS in a row. Initially, we have C=0, A = 1. After first MPS, A becomes 0.55, and re-normalized to 1.1, while C remains 0. After second MPS, A becomes 0.65, and re-normalized to 1.3, while C remains 0. After third MPS, A becomes 0.85, and no re-normalization is performed. When we encode the fourth MPS, the sub-interval allocated to the MPS becomes A – Qe = 0.40, while the sub-interval allocated to LPS is 0.45. Now, the sub-interval allocated to MPS is smaller than the LPS sub-interval. It can be seen that the condition for this situation is that Qe > A / 2, or Qe > A - Qe. The solution is to perform *conditional exchange*: interchanging the two intervals under the afore-mentioned condition. Because conditional exchange is caused by the approximated re-normalization, the test for conditional exchange is only performed when re-normalization is needed.

The pseudo-code for the approximated re-normalization with conditional exchange is:

```
When MPS is encoded
        begin
            C is unchanged;
            A: = A – Qe;
            If (A<0.75) then
                If (A<Qe) then
                        C:=C+A
                        A = Qe
                Endif
            Renormalize A and C;
            Endif;
        End
When LPS is encoded
        begin
            If (A-Qe>=Qe) then
                C:=C+A - Q
                A = Qe
            Else
                A = A-Qe
            Endif
            Renormalize A and C;
        End
```