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Simplified robust control for nonlinear uncertain systems: a method of projection and online estimation☆

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Abstract

Robust control based on an online estimation of uncertainty is presented for a class of nonlinear uncertain systems. The estimation is done via a robust observer after the uncertainty vector is projected onto a one-dimensional subspace. The proposed combination of dynamics projection and online estimation is to relax the knowledge about the size of uncertainty and required in the robust control design, to make robust control less conservative while being effective, and to ensure robust stability without undue complexity.

Keywords: Nonlinear robust control; Subspace projection; Uncertainty estimation

1. Problem statement and introduction

The class of uncertain systems considered in the paper is of form

$$\dot{x} = f(x,t) + B(x,t)[\Delta f(x,t) + u], \tag{1}$$

where $x \in \Re^n$ is the state, $f(x,t)$ and $B(x,t)$ are the known dynamics, and $\Delta f(x,t) \in \Re^m$ is a lumped and matched uncertainty vector bounded as, for all (x,t) and for some scalar, nonnegative and locally uniformly bounded but *unknown* function $\rho(\|x\|)$,

$$\|\Delta f(x,t)\| \leqslant \rho(\|x\|), \quad \forall x \in \Re^n. \tag{2}$$

Due to the presence of uncertainties, control $u \in \Re^m$ must be designed to robustly compensate for their influence in dynamics.

Control of nonlinear uncertain systems has been an active research area for decades. Among various design approaches, adaptive control and robust control are most popular, and both are capable of dealing with the unknowns. In simple terms, an adaptive scheme is to estimate unknown but constant parameters in system dynamics via adaptation laws (Kanellakppoulos, Kokotovic, & Morse, 1991; Krstic, Kanellakppoulos, & Kokotovic, 1995; Ioannou & Sun, 1995), and robust control is to stabilize an uncertain system by assuming that its uncertainties be bounded in size by a known function (Corless & Leitmann, 1981; Freeman & Kokotovic, 1996; Qu, 1998).

To guarantee stability, a nonlinear robust control is typically designed based on bounding function $\rho(\cdot)$ in (2) as compensation by the robust control must dominate the effect of uncertainties. While the domination is necessary for robustness, it has been achieved technically in the existing results of robust control design by assuming either the exact knowledge of $\rho(\cdot)$ or its functional expression (so an adaptive robust control (Corless & Leitmann, 1983) can be

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devised). In both cases, the uncertainty $\Delta f(x,t)$ is first bounded, and the design is based on its bounding function. Recently, an extension is made in Qu and Jin (2001) so that, if uncertainties are generated by certain exogenous systems, robust control can be designed without explicit knowledge of their bounding function.

In this paper, a new approach is proposed and it involves two key steps. First, the lumped uncertainty is projected onto the subspace of the gradient of a proper Lyapunov function together with input matrix of the system. It is shown that, for robust stability, the scalar projection of uncertainty is the only term to be estimated and compensated for, not the uncertainty itself. This motivates us to design a first-order robust observer which renders a dynamic robust control of the lowest order. Second, it is shown that, if the projection of uncertainty is differentiable and the size of its time derivative can be related to the size of uncertainty projection, robust estimation and control design can be done without requiring any information of bounding function on uncertainty or its projection, either magnitude(s) or functional expression(s). A constructive algorithm is included to show how the size of the time derivative of uncertainty projection can be related to the unknown size of the projection. Thus, compared to the existing adaptive and robust controls, the proposed control is both much simpler and less conservative. It is also shown that the robust observer is equivalent to an auxiliary output passing through a first-order high-pass filter, where the auxiliary output is an integral manifold of uncertainty impact in the time derivative of a proper Lyapunov function along system trajectories.

2. Technical conditions

In order to design a robustly stabilizing control for system (1) without the knowledge of $\rho(\cdot)$, the following assumptions are introduced. The first assumption is on the whole system dynamics so that, under any continuous control, existence of a classical solution to differential equation (1) is guaranteed (Khalil, 1996).

Assumption 1. All functions in (1) are continuous, uniformly bounded with respect to t, and locally uniformly bounded with respect to x.

Robust control design is usually carried out by first investigating stability or stabilizability of the corresponding nominal system, denoted by $\dot{x}=f(x,t)+B(x,t)u$. If one cannot find a stabilizing control for the known nominal system, there is little hope that a robust control can be found. This means that, for the purpose of robust control design, the following assumption is needed technically.

Assumption 2. The origin is a globally asymptotically stable equilibrium point for the uncontrolled nominal system, $\dot{x}=f(x,t)$. Furthermore, Lyapunov function $V(x,t)$, guaranteed by the Lyapunov converse theorem (Khalil, 1996), is found such that $\gamma_1(\|x\|)\leqslant V(x,t)\leqslant\gamma_2(\|x\|)$ and $\partial V/\partial t+\nabla_x^{\mathrm{T}}V(x,t)f(x,t)\leqslant-\gamma_3(\|x\|)$, where $\gamma_1(\cdot)$, $\gamma_2(\cdot)$, and $\gamma_3(\cdot)$ are class $\mathscr{K}_\infty$ functions.

In this paper, robust control is designed without the standard assumption that bounding function $\rho(x,t)$ is known or that the uncertainty $\Delta f(x,t)$ or its bounding function is parameterizable. Our approach is to estimate online a scalar projection of the uncertainty. Given Lyapunov function $V(x,t)$ and any vector function $g(x,t)\in\Re^n$, the projection of $g(x,t)$ along the gradient of $V(x,t)$ is defined by

$$P_{V(x,t)}^{g(x,t)}\triangleq\nabla_x^{\mathrm{T}}V(x,t)g(x,t),$$

which is also the directional derivative of $V(x,t)$ along the vector field of $g(x,t)$. To be specific, scalar projection $P_{V(x,t)}^{B(x,t)\Delta f(x,t)}$ of uncertainty will be estimated. Let the estimation error be denoted by e. Then, the objective of robust control problem can be characterized by making x and e converge according to sets $\Omega_x^*=\{x\in\Re^n:\|x\|\leqslant\varepsilon^*\}$ and $\Omega_e^*=\{e\in\Re:|e|\leqslant\varepsilon^*\}$, where constant $\varepsilon^*>0$ can be selected from accuracy requirement. To ensure observability and observer-based robust stability, the following assumption is introduced.

Assumption 3. Uncertainty $\Delta f(x,t)$ has the property that its scalar projection, $P_{V(x,t)}^{B(x,t)\Delta f(x,t)}$, is differentiable and satisfies the following differential inequality along trajectories of system (1):

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}P_{V(x,t)}^{B(x,t)\Delta f(x,t)}\right|\leqslant\alpha_1(x,t)|P_{V(x,t)}^{B(x,t)\Delta f(x,t)}|^{\beta_1}+\alpha_2(x,t)\times\|u\|^{\beta_2}+\alpha_3(x,t),\quad\forall x\notin\Omega_x^*,$$

where $\beta_i\geqslant 1$ are constants, and scalar functions $\alpha_i(x,t)$ are known, locally uniformly bounded with respect to x, and uniformly bounded with respect to t.

Implications and establishment of Assumption 3 will be elaborated at the end of the subsequent section.

3. Design of the proposed robust control

Estimation of nonlinear uncertainty has been pursued in two ways: (1) Develop a bounding function on the size of the uncertainty, linearly parameterize the bounding function (or the uncertainty directly) in terms of a number of unknown parameters, and estimate the parameters using adaptation laws. (2) If the uncertainties are generated as the output of some exogenous system, they can be estimated and compensated under certain conditions. In either of these two cases, certain structure property about the uncertainty or its bounding function is needed. To relax such required structural information, direct estimation of the uncertainty needs to be done as is. Among options of directly estimating the uncertainty, we choose to do so through a Lyapunov argument. This is motivated by the fact that stability analysis and control design can always be done using the Lyapunov direct method, including the domination concept. In fact, by considering effect of uncertainty in stability, one can map the uncertainty into the subspace where its effect in the Lyapunov argument can be quantified and estimated. To this end, define the so-called null set

$$\mathcal{N} = \{x \in \Re^n : \nabla_x^{\mathrm{T}} V(x,t) B(x,t) = 0\}.$$

Now, consider the time derivative of Lyapunov function $V(x,t)$ along trajectories of system (1), that is,

$$\dot{V}(x,t) = \frac{\partial V(x,t)}{\partial t} + \nabla_x^{\mathrm{T}} V(x,t) f(x,t) + \nabla_x^{\mathrm{T}} V(x,t) B(x,t) u + P_{V(x,t)}^{B(x,t)\Delta f(x,t)}. \tag{3}$$

It is obvious that the impact of uncertainty $\Delta f(x,t)$ in $\dot{V}$ is the projection of $P_{V(x,t)}^{B(x,t)\Delta f(x,t)}$. The projection is active (i.e., nonzero) only in the complement of set $\mathcal{N}$, denoted by $\mathcal{N}^{\mathrm{c}}$ or by quotient set $\Re^n/\mathcal{N}$. To estimate the impact, we can define

$$\xi(t) \triangleq V(x,t) - V(x_0,t_0) - \int_{t_0}^{t} \left[\nabla_x^{\mathrm{T}} V(x,t)(f(x,t) + B(x,t)u) + \frac{\partial V(x,t)}{\partial t} \right] \mathrm{d}t \tag{4}$$

to be an auxiliary output that could be calculated by definition. It follows that

$$\dot{\xi}(t) = \nabla_x^{\mathrm{T}} V(x,t) B(x,t) \Delta f(x,t) = P_{V(x,t)}^{B(x,t)\Delta f(x,t)}, \tag{5}$$

which is the impact term. Thus, the problem of estimating uncertainty's impact in $\dot{V}$ becomes the problem of estimating $\dot{\xi}(t)$ from $\xi(t)$, and the latter could be accomplished using a high-pass filter as shown in the following lemma. Since $\dot{\xi}(t) \in \Re$ and $\Delta f(x,t) \in \Re^m$, the advantage of estimating $\dot{\xi}(t)$ rather than $\Delta f(x,t)$ directly is obvious.

Lemma 1. *Suppose that system* (1) *satisfies Assumptions* 1 *and* 2. *Consider the estimator*

$$\mu\dot{\eta} = -\eta - \frac{1}{\mu}\xi(t), \quad w(t) = \frac{1}{\mu}\xi(t) + \eta, \tag{6}$$

where $\xi(t)$ is the auxiliary output in (4), *$\mu > 0$ is a design parameter, and initial condition $\eta(t_0)$ is chosen such that $\eta(t_0) = c_0\|\nabla_x^{\mathrm{T}} V(x(t_0),t_0) B(x(t_0),t_0)\|$ for some constant $c_0 \in \Re$. Then, for all sufficiently small values of $\mu > 0$:*

- *If $x \to \mathcal{N}$, $\|\nabla_x^{\mathrm{T}} V(x,t) B(x,t)\| \to 0$, and $w(t)$ converges to zero at the same rate.*
- *If $\|\ddot{\xi}(t)\|$ is bounded by a constant, estimation error $e(t) \triangleq [\dot{\xi}(t) - w(t)]$ is uniformly bounded and uniformly ultimately bounded.*
- *If $\|\ddot{\xi}(t)\|$ converges to zero, so will estimation error $e(t)$.*

Proof. Taking time derivative of the filter output in Eq. (6) yields

$$\dot{w} = \frac{1}{\mu}\dot{\xi}(t) - \dot{\eta} = \frac{1}{\mu}\dot{\xi}(t) - \frac{1}{\mu}\eta - \frac{1}{\mu^2}\xi(t),$$

or simply,

$$\mu\dot{w} = \dot{\xi}(t) - w(t). \tag{7}$$

It is obvious from (5) and (7) that, if x is in the vicinity of $\mathcal{N}$ and converges into it, $w(t)$ will converge to zero and that, for sufficiently small μ and with $w(t_0) = \eta(t_0)$, convergence of $w(t)$ is the same as that of $\|\nabla_x^{\mathrm{T}} V(x,t) B(x,t)\|$. Thus, the ratio

$$\frac{\|w(t)\|}{\|\nabla_x^{\mathrm{T}} V(x,t) B(x,t)\|}$$

can be maintained to be finite.

To show the other properties of estimation error, define $e(t) = \dot{\xi}(t) - w(t)$ and note from Eq. (7) that

$$\frac{\mathrm{d}e^2(t)}{\mathrm{d}t} = 2e(t)\dot{e}(t) = -\frac{2}{\mu}\mathrm{e}^2(t) + 2e(t)\ddot{\xi}(t).$$

Hence, the last two statements of the lemma can be concluded using the above equation. □

Introduction of quantity $\xi(t)$ in (4) and high-pass filter (6) is very useful in motivating our basic idea and formulating the proposed design. In fact, the high-pass filter has transfer function $s/(\mu s + 1)$ and is commonly used. The difference here is that an unstructured differentiable unknown within a Lyapunov argument is estimated. However, a direct implementation of (4) and (6) has the shortcoming that, in (4), there is a pure integrator operated over an infinite horizon (as t increases). This problem can be overcome because, as shown in the proof of Lemma 1, impact of the uncertainty is estimated by w, not by $\xi(t)$ or $\eta(t)$. It follows from (7), (5), (3) and (6) that $w(t)$ can be calculated directly from the following observer:

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mu w - V(x,t)] = -w - \nabla_x^{\mathrm{T}} V(x,t)(f(x,t) + B(x,t)u) - \frac{\partial V(x,t)}{\partial t}, \tag{8}$$

where $w(t_0) = c_0\|\nabla_x^{\mathrm{T}} V(x(t_0),t_0) B(x(t_0),t_0)\|$. Clearly, all the properties in Lemma 1 are maintained under (8). Robust observer (8) is well defined and, while there are nonlinear functions of x, it contains a high-pass filter from $V(x,t)$ to $w(t)$ and a low-pass filter from the rest of the terms to $w(t)$.

Since nonlinear systems do not generally observe the separation principle, the proposed scheme of estimating uncertainty impact online by robust estimator (8) must be analyzed as a part of an overall robust control design. It follows from (3) that, if auxiliary output $w(t)$ serves as a good estimate of uncertainty effect $\dot{\xi}(t)$ in (5), the following should be a robust control candidate:

$$u_{\mathrm{r}}(x,t,w) = -\frac{B^{\mathrm{T}}(x,t)\nabla_x V(x,t)}{\|\nabla_x^{\mathrm{T}} V(x,t) B(x,t)\|^2}\, w(t), \tag{9}$$

where $w(t)$ is given by (8). In this sense, the above robust control design can be viewed as an extension of the certainty equivalence principle. Property of the above robust control candidate and its closed-loop stability result is summarized by the following theorem. In its proof, inequality (12) is obtained for the purpose of making an explicit selection of design parameter μ.

Theorem. *Consider system* (1) *under Assumptions* 1–3. *Then, the robust control in* (9) *and* (8) *ensures that, for sufficiently small* $\mu > 0$, *the closed loop system is either uniformly ultimately bounded with respect to any accuracy level* (*specified by* ε^*) *or, if* $\ddot{\xi} \to 0$ *as* $x \to 0$, *asymptotically stable. And, the stability results are semi-global.*

Proof. Consider compact sets $\Omega_x = \{x \in \Re^n : \|x\| \leqslant c_x\}$ and $\Omega_e = \{e \in \Re : |e| \leqslant c_e\}$, where c_x and c_e are positive but otherwise arbitrarily chosen constants. And, let $\|\cdot\|_\tau$ be the truncated functional norm defined by $\|y(t)\|_\tau \triangleq \sup_{t_0 \leqslant t \leqslant t_0+\tau} \|y(t)\|$. Now, suppose that, for some $\tau > 0$, $\|x\|_\tau \in \Omega_x$ and $|e|_\tau \in \Omega_e$. Stability analysis is done in four steps.

First, it follows from Lemma 1 that, although its denominator contains a second-order term related to set $\mathcal{N}$, the robust control in (9) is locally uniformly bounded. That is, there is constant c_u such that $\|u\|_\tau \leqslant c_u$. In addition, it follows from (8) that $\|w\|_\tau \leqslant c_w$ for some constant c_w.

Second, given $\|x\|_\tau \in \Omega_x$, there exist constants $\lambda_{\alpha_i} \geqslant 0$ ($i = 1, 2, 3$) such that $|\alpha_i(x,t)|_\tau \leqslant \lambda_{\alpha_i}$. Therefore, we know from Assumption 3 that

$$\left|\ddot{\xi}\right|_\tau \leqslant \lambda_{\alpha_1} \left|\dot{\xi}\right|_\tau^{\beta_1} + \lambda_{\alpha_2} c_u^{\beta_2} + \lambda_{\alpha_3}.$$

As the third step, consider Lyapunov function

$$L(x,t) = V(x,t) + \tfrac{1}{2}[\dot{\xi}(t) - w]^2 = V(x,t) + \tfrac{1}{2}|e|^2,$$

which is globally positive definite and radially unbounded. Therefore, there exist class-$\mathcal{K}_\infty$ functions $\gamma_4, \gamma_5 : \Re^+ \to \Re^+$ such that $\gamma_4(\|\Psi\|) \leqslant L(x,e,t) \leqslant \gamma_5(\|\Psi\|)$, where $\Psi = [x^{\mathrm{T}}\ e]^{\mathrm{T}}$. Note that $w(t_0)$ is independent of μ and so are initial conditions $x(t_0)$ and $\dot{\xi}(t_0)$. In what follows, a Lyapunov argument will be carried out by considering all initial conditions in the following set:

$$\Omega_{\Psi(0)} \triangleq \{\Psi(t_0) : \|\Psi(t_0)\| \leqslant \gamma_5^{-1} \circ \gamma_4(\sqrt{c_x^2 + c_e^2})\}. \tag{10}$$

Finally, given the sets $\Omega_x^* = \{x \in \Re^n : \|x\| \leqslant \varepsilon^*\}$ and $\Omega_e^* = \{e \in \Re : |e| \leqslant \varepsilon^*\}$, it follows from (1), (9), and (7) that, given $\|x\|_\tau \in \Omega_x$ and given initial conditions specified in (10),

$$\begin{aligned}
\dot{L}(x,t) &= \frac{\partial V}{\partial t} + \nabla_x^{\mathrm{T}} V f + (\dot{\xi} - w) \\
&\quad + (\dot{\xi} - w)\left[\ddot{\xi} - \frac{1}{\mu}(\dot{\xi} - w)\right] \\
&\leqslant -\gamma_3(\|x\|) - \frac{1}{\mu}|e|^2 + |e| \\
&\quad \times [\lambda_{\alpha_1} |\dot{\xi}|_\tau^{\beta_1} + \lambda_{\alpha_2} c_u^{\beta_2} + \lambda_{\alpha_3} + 1] \\
&\leqslant -\gamma_3(\|x\|) - \frac{1}{\mu}|e|^2 + |e| \\
&\quad \times [\lambda_{\alpha_1}[(|e|_\tau + c_w)^{\beta_1} - c_w^{\beta_1}] + \lambda_{\alpha_1} c_w^{\beta_1} \\
&\quad + \lambda_{\alpha_2} c_u^{\beta_2} + \lambda_{\alpha_3} + 1] \\
&\leqslant -\gamma_3(\|x\|) - \frac{1}{3\mu}|e|^2
\end{aligned} \tag{11}$$

holds for all $x \in \Omega_x / \Omega_x^*$ and $e \in \Omega_e / \Omega_e^*$ provided that μ is chosen to satisfy

$$0 < \mu < \min\left\{ \frac{\varepsilon^*}{3(\lambda_{\alpha_1} c_w^{\beta_1} + \lambda_{\alpha_2} c_u^{\beta_2} + \lambda_{\alpha_3} + 1)},\ \frac{c_e}{3\lambda_{\alpha_1}[(c_e + c_w)^{\beta_1} - c_w^{\beta_1}]} \right\}. \tag{12}$$

According to Theorem 2.15 in Qu (1998), subvectors in augmented state vector Ψ will be uniformly bounded (with respect to their own sets within $\Omega_{\Psi(0)}$ in (10)) and uniformly ultimately bounded (with respect to a hyperball whose radius is a class-$\mathcal{K}$ function of ε^*).

It follows from (5) and (8) (or (7)) that, as $\|x\|$ approaches zero, so will be $\dot{\xi}$ and hence w. If $\ddot{\xi}$ has a similar property (which is usually not known a priori), it follows from (11) and from the third statement in Lemma 1 that the closed-loop system is asymptotically stable.

Since bounds such as c_x and c_e are arbitrary, independent of μ, and can be increased, the closed loop stability results are semi-global. $\square$

The theorem is concluded under the newly introduced condition, Assumption 3. To fully justify its merits, it is necessary to compare Assumption 3 against the standard assumption of inequality (2). The proof of the following lemma provides a direct and constructive algorithm to establishing Assumption 3 (by finding functions $\alpha_i(\cdot)$).

Lemma 2. *Consider system* (1) *with* $m = 1$. *Suppose that, for some known functions* $q_i(\cdot)$ *and constants* c_{q_i} *such that*

$$\begin{aligned}
\left|\frac{\partial \Delta f(x,t)}{\partial t}\right| &\leqslant q_1(x) + |\Delta f(x,t)|^{c_{q_1}}, \\
\left\|\frac{\partial \Delta f(x,t)}{\partial x}\right\| &\leqslant q_2(x) + |\Delta f(x,t)|^{c_{q_2}},
\end{aligned} \tag{13}$$

and that, for some function $\rho'(\cdot)$,

$$\|\Delta f(x,t)\| \leqslant \rho'(x) \quad \forall x \in \mathcal{N}_{\varepsilon^*}/\Omega_x^*, \tag{14}$$

where $\mathcal{N}_{\varepsilon^*} \triangleq \{x \in \Re^n : \|\nabla_x^{\mathrm{T}} V(x,t)B(x,t)\| \leqslant \varepsilon^*\}$. Then, functions $\alpha_1(x)$, $\alpha_2(x)$ and $\alpha_3(x)$ in Assumption 3 can always be found.

Proof. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{V(x,t)}^{B(x,t)\Delta f(x,t)} = \frac{\partial P_{V(x,t)}^{B(x,t)\Delta f(x,t)}}{\partial t} + \left[\frac{\partial P_{V(x,t)}^{B(x,t)\Delta f(x,t)}}{\partial x}\right]^{\mathrm{T}} \dot{x}.$$

Therefore, we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t} P_{V(x,t)}^{B(x,t)\Delta f(x,t)}\right| \leqslant & \left|\frac{\partial P_{V(x,t)}^{B(x,t)}}{\partial t}\right| \cdot |\Delta f(x,t)| + |P_{V(x,t)}^{B(x,t)}| \\ & \cdot \left|\frac{\Delta f(x,t)}{\partial t}\right| + \left[\left\|\frac{\partial P_{V(x,t)}^{B(x,t)}}{\partial x}\right\|\right. \\ & \cdot |\Delta f(x,t)| + |P_{V(x,t)}^{B(x,t)}| \\ & \cdot \left.\left\|\frac{\Delta f(x,t)}{\partial x}\right\|\right] [|f(x,t)| + |B(x,t)| \\ & \cdot |\Delta f(x,t)| + |B(x,t)| \cdot |u|]. \end{aligned} \tag{15}$$

After substituting the expressions in (13) into (15), functions $\alpha_i(x)$ can be found using the Holder's inequality to validate Assumption 3. Specifically, for $x \in \mathcal{N}_{\varepsilon^*}/\Omega_x^*$, the magnitude of $P_{V(x,t)}^{B(x,t)}$ is small but $|\Delta f(x,t)|$ is already bounded as given in (14); and for $x \in \Re^n/(\mathcal{N}_{\varepsilon^*} \cup \Omega_x^*)$, the relationship of $|\Delta f(x,t)| = |P_{V(x,t)}^{B(x,t)}|^{-1} \cdot |P_{V(x,t)}^{B(x,t)\Delta f(x,t)}|$ can be used. □

It is worth noting that the inequalities in (13) can be justified if the lumped uncertainty $\Delta f(x,t)$ is differentiable. Compared to the standard condition of (2), condition (14) is *relaxed* as the size bounding function is only needed in the set $\mathcal{N}_{\varepsilon^*}/\Omega_x^*$ rather than the whole state space. In the current setting, condition (14) is needed to prevent the trajectory from becoming unbounded within $\mathcal{N}/\Omega_x^*$ since, in set $\mathcal{N}$, uncertainty impact measured by the chosen Lyapunov function is little and hence uncertainty estimation is ineffective therein. However, as shown in Qu and Dorsey (1992), Lyapunov functions are not unique and control, as well as estimation of uncertainty, does not have to be limited to the use of one Lyapunov function. In other words, it is possible to remove condition (14) in general by estimating uncertainty along projections of several appropriate Lyapunov functions, which is a subject beyond the scope of this correspondence.

It is also worth noting that, for scalar systems with $B(x,t) = 1$, Lyapunov function can be chosen to be $V(x,t) = 0.5x^2$. In this case, set $\mathcal{N}_{\varepsilon^*}/\Omega_x^*$ in condition (14) is empty. That is, condition (14) is not needed for most scalar systems. Furthermore, for any *n*-dimensional uncertain system satisfying the strict feedback structure in which $B = [0 \cdots 0\, 1]^{\mathrm{T}}$ (e.g., the subsequent example), it is straightforward to show using the backstepping procedure that stabilization of the overall system boils down to uncertainty estimation and robust control of its bottom scalar subsystem. Therefore, for this class of uncertain systems, Lemma 2 again applies without the need of imposing condition (14). In other words, at least one class of uncertain systems are found for which robust stabilization does not require that the size of either the uncertainty or its projection (or their time/partial derivatives) be known.

In summary, the theorem and Lemma 2 together show that, although the uncertainty is unstructured and its size bounding function is unknown nor parameterizable, its impact on system dynamics can be online estimated by a scalar robust estimator. It is encouraging that robust control can be designed without assuming a usually conservative size bounding function or using a pure integrator (such as adaptive control or some adaptive robust control). And, the resulting estimation-based robust control is simpler, less conservative, and semi-globally stabilizing.

4. An illustrative example

Consider the following uncertain system:

$$\dot{x}_1 = x_1 + (1 + x_1^2)x_2 \quad \text{and} \quad \dot{x}_2 = x_1 + \Delta f + v.$$

It is straightforward to verify that the nominal system is feedback linearizable with respect to the output–input pair of $\{x_1, v\}$. Using the backstepping procedure, one can obtain Lyapunov function $V(x) = 0.5[x_1^2 + (2x_1 + b(x_1)x_2)^2]$ and its corresponding nominal stabilizing control

$$u_n(x) \triangleq -\frac{x_1 + (3 + 2x_1x_2)(x_1 + b(x_1)x_2)}{b(x_1)} - x_1 \tag{16}$$

so that Assumption 2 is satisfied. It follows that $\nabla_x^{\mathrm{T}} V(x)B(x) = 2b(x_1)x_1 + b^2(x_1)x_2$ with $b(x_1) = 1 + x_1^2$. Hence, the proposed first-order dynamic robust control is $v = u_n(x) + u_r(x,w)$, where $u_n(x)$ is given by (16), $u_r(x,w)$ is described by (9), and $w(t)$ is defined by (8). In particular, the impact of uncertainty project in $\dot{V}$ is estimated by $w(t)$, and $w(t)$ generated by robust observer (8) is given by

$$\begin{aligned} w(t) &= \frac{1}{\mu}\varphi(t) + \frac{1}{\mu}V(x) \\ &= \frac{1}{\mu}\varphi(t) + \frac{1}{2\mu}[x_1^2 + (2x_1 + b(x_1)x_2)^2] \end{aligned}$$

and

$$\begin{aligned} \dot{\varphi}(t) &= -\frac{1}{\mu}\varphi(t) - \frac{1}{\mu}V(x) - \nabla_x^{\mathrm{T}} V(x)[f(x) + B(x)v] \\ &= -\frac{1}{\mu}\varphi(t) - \frac{1}{2\mu}[x_1^2 + (2x_1 + b(x_1)x_2)^2] \\ &\quad - x_1(x_1 + b(x_1)x_2) - (2x_1 + b(x_1)x_2) \\ &\quad \times [b(x_1)x_1 + 2(1 + x_1x_2)(x_1 + b(x_1)x_2) + b(x_1)v]. \end{aligned}$$

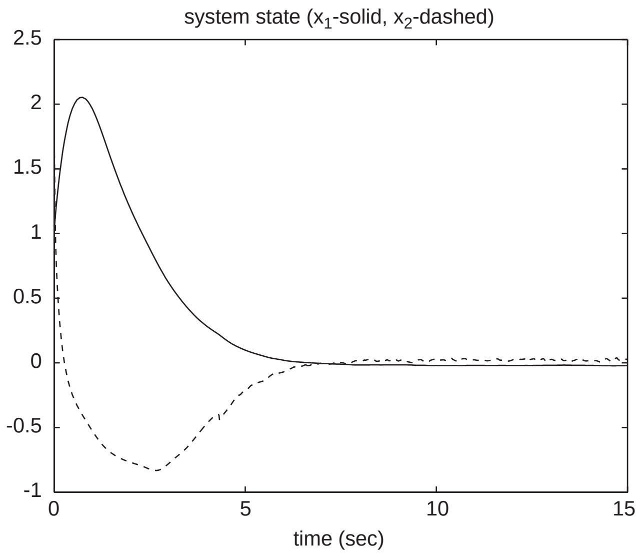

Fig. 1. State variables x_1 and x_2.

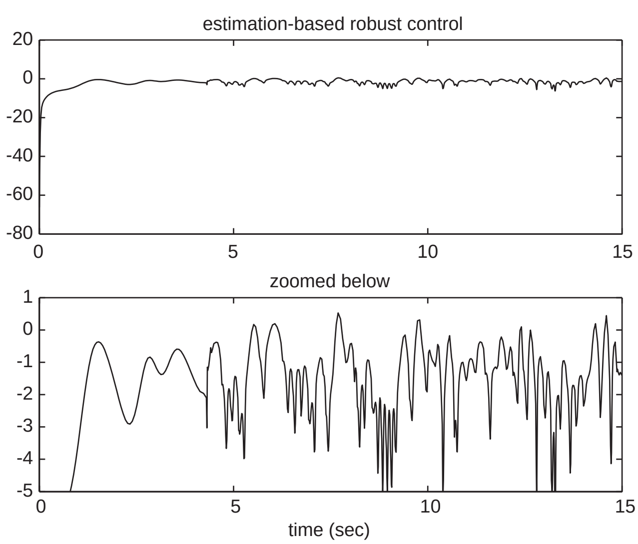

Fig. 2. Estimation-based robust control $u(t)$.

In the simulation, the design parameter is set to be $\mu = 0.015$, initial conditions are selected to be $x(0) = [2\ 1]^{\mathrm{T}}$ and $w(0) = -10$, and the "uncertainty" is chosen to be

$$\Delta f(x,t) = -\sin(\pi t)\cos(2\pi(\sin(2\pi t/5)+0.5)t/4)+3x_1x_2 \\ +2x_1+\cos(2\pi t/2.5)\sin(2\pi t/5)+1.$$

Simulation results of the closed-loop system are shown in Figs. 1 and 2.

5. Conclusion

Robust control of a class of nonlinear uncertain systems is pursued by on-line estimating unstructured nonlinear uncertainty rather than assuming the knowledge of their size bounding functions. It is shown that, through projection, the effect of uncertainty on stability can be estimated by the output of a scalar robust estimator. Such an estimation can always be done under the condition that the uncertainty projection is differentiable and the size of its time derivative can be related back to the size of the projection. And, an observer-based robust control is proposed to achieve semi-global stability.

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