## CDA6530: Performance Models of Computers and Networks

# Chapter 6: Elementary Queuing Theory

# Definition

- Queuing system:
  - a buffer (waiting room),
  - service facility (one or more servers)
  - a scheduling policy (first come first serve, etc.)
- We are interested in what happens when a stream of customers (jobs) arrive to such a system
  - throughput,
  - sojourn (response) time,
    - Service time + waiting time
  - number in system,
  - server utilization, etc.

# *Terminology*

- A/B/c/K queue
  - A - arrival process, interarrival time distr.
  - B - service time distribution
  - c - no. of servers
  - K - capacity of buffer

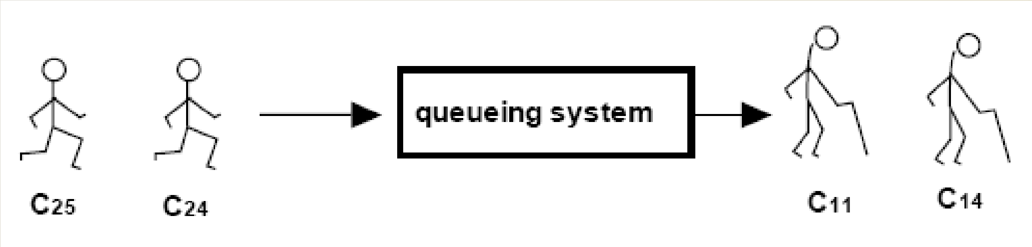
  - Does not specify scheduling policy

# *Standard Values for A and B*

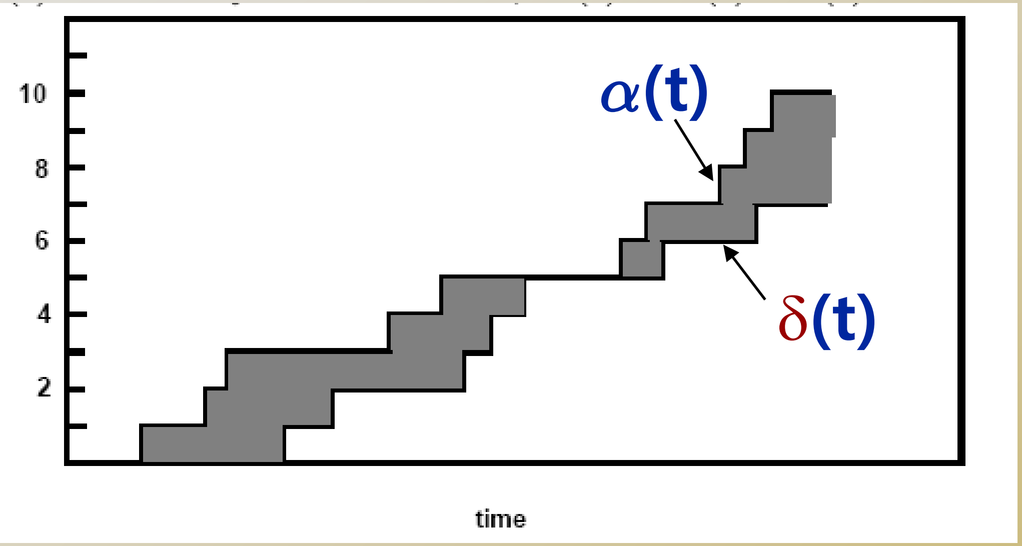
- M - exponential distribution (M is for Markovian)
- D - deterministic (constant)
- GI; G - general distribution

- M/M/1: most simple queue
- M/D/1: expo. arrival, constant service time
- M/G/1: expo. arrival, general distr. service time

# Some Notations




- $C_n$: custmer n, n=1,2,$\cdots$
- $a_n$: arrival time of $C_n$
- $d_n$: departure time of $C_n$
- $\alpha(t)$: no. of arrivals by time t
- $\delta(t)$: no. of departure by time t
- N(t): no. in system by time t
  - $N(t)=\alpha(t)-\delta(t)$




- Average arrival rate (from t=0 to now):
  - $\lambda_t = \alpha(t)/t$

# Little's Law

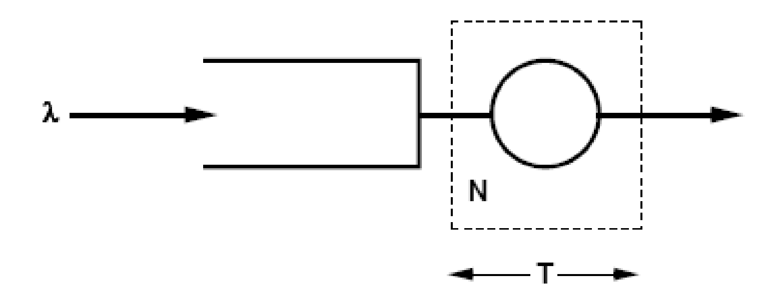
- $\gamma(t)$: total time spent by all customers in system during interval (0, t)

$$\gamma(t) = \sum_{n=1}^{\alpha(t)} \min\{d_n, t\} - a_n = \int_0^t N(s)ds$$

- $T_t$: average time spent in system during (0, t) by customers arriving in (0, t) $T_t = \gamma(t)/\alpha(t)$
- $N_t$: average no. of customers in system during (0, t)
  - $N_t = \gamma(t)/t$

- For a stable system, $N_t = \lambda_t T_t$
  - Remmeber $\lambda_t = \alpha(t)/t$
- For a long time and stable system
- $N = \lambda T$
- Regardless of distributions or scheduling policy

# *Utilization Law for Single Server Queue*

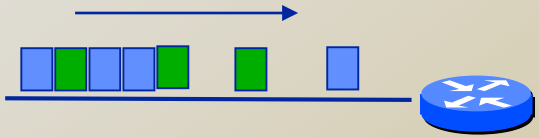


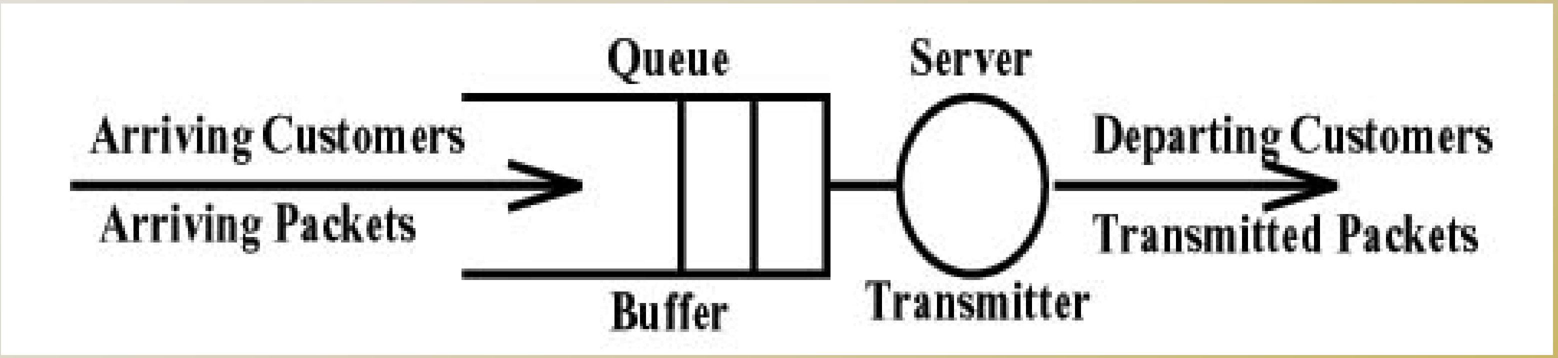

- X: service time, mean T=E[X]
- Y: server state, Y=1 busy, Y=0 idle
- $\rho$: server utilization, $\rho = P(Y=1)$
- Little's Law: $N = \lambda E[X]$
- While: $N = P(Y=1)\cdot 1 + P(Y=0)\cdot 0 = \rho$
- Thus Utilization Law:

$$\rho = \lambda E[X]$$

Q: What if the system includes the queue?

# *Internet Queuing Delay Introduction*





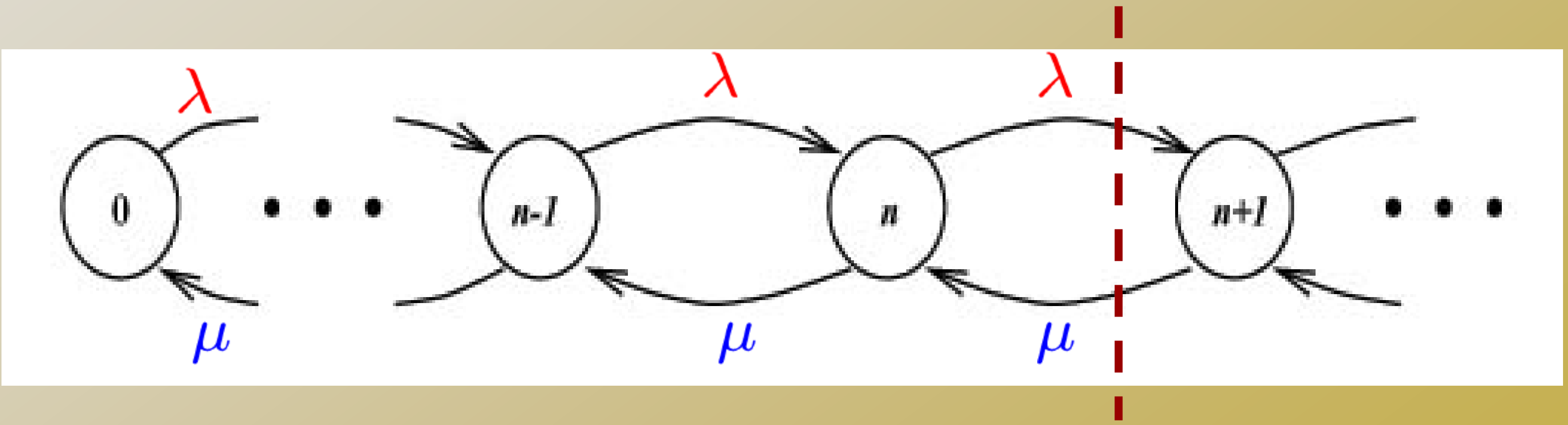

- How many packets in the queue?
- How long a packet takes to go through?

# The M/M/1 Queue

- An M/M/1 queue has
  - Poisson arrivals (with rate λ)
    - Exponential time between arrivals
  - Exponential service times (with mean 1/μ, so μ is the "service rate").
  - One (1) server
  - An infinite length buffer
- The M/M/1 queue is the most basic and important queuing model for network analysis

# State Analysis of M/M/1 Queue

- N : number of customers in the system
  - (including queue + server)
  - Steady state
- $\pi_n$ defined as $\pi_n$=P(N=n)
- $\rho=\lambda/\mu$: Traffic rate (traffic intensity)

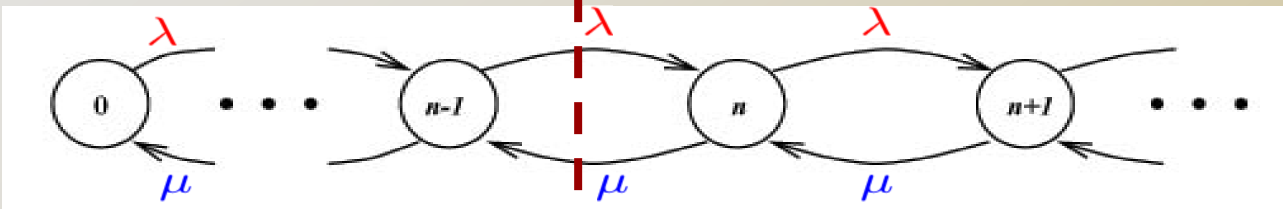



State transition diagram

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu & -(\lambda+\mu) & \lambda & \cdots \\ 0 & \mu & -(\lambda+\mu) & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

- we can use $\pi$Q = 0 and $\sum \pi_i = 1$
- We can also use balance equation

# *State Analysis of M/M/1 Queue*




- # of transitions → = # of transitions ←

$$\pi_0 \lambda = \pi_1 \mu \quad \Rightarrow \pi_1 = \rho \pi_0$$

$$\pi_1 \lambda = \pi_2 \mu \quad \Rightarrow \pi_2 = \rho^2 \pi_0$$

$$\vdots \qquad\qquad \vdots$$

$$\pi_{n-1} \lambda = \pi_n \mu \quad \Rightarrow \pi_n = \rho^n \pi_0$$

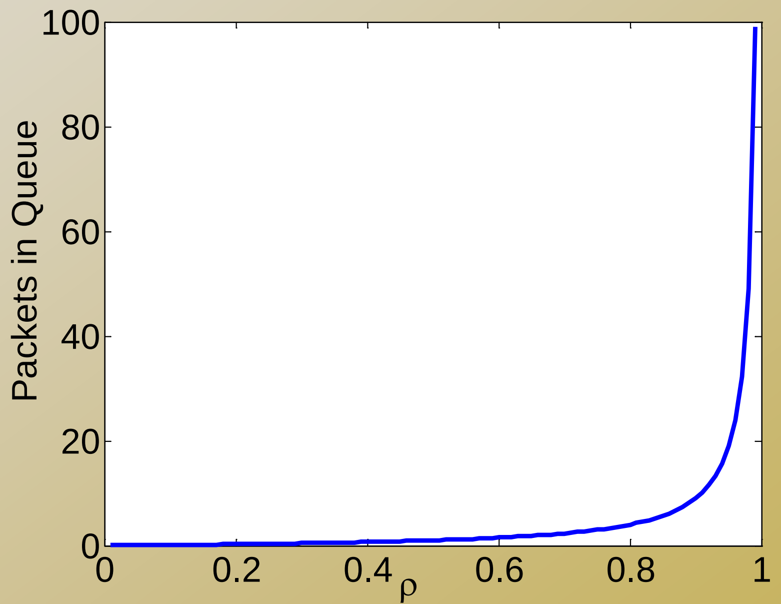
$\pi_n$ are probabilities: $\sum_{i=0}^{\infty} \pi_i = 1$ $\quad \Rightarrow \quad \pi_0 = 1 - \rho$

$\rho = 1 - \pi_0$ : prob. the server is working (why ρ is called **"server utilization"**)

# State Analysis of M/M/1 Queue

- N: avg. # of customers in the system

$$E[N] = \sum_{k=1}^{\infty} k\pi_k = \pi_0 \sum_{k=1}^{\infty} k\rho^k = \frac{\rho}{1-\rho}$$



# *M/M/1 Waiting Time*

- $X_n$: service time of n-th customer, $X_n =_{st} X$ where X is exponential rv
- $W_n$: waiting time of n-th customer
  - Not including the customer's service time
- $T_n$: sojourned time $T_n = W_n + X_n$
- When $\rho < 1$, steady state solution exists and $X_n, W_n, T_n \rightarrow X, W, T$
- 
- Q: E[W]?

# State Analysis of M/M/1 Queue

- W: waiting time for a new arrival

$$W = X_1 + X_2 + \cdots + X_{n-1} + R$$

$X_i$ : service time of i-th customer

$R$ : remaining service time of the customer in service
Exponential r.v. with mean 1/μ due to **memoryless** property of expo. Distr.

$$E[W] = E[(N-1)X] + E[R] = E[N] \cdot E[X]$$

- T: sojourn (response) time

$$E[T] = \frac{1}{\mu} + E[W] = \frac{1}{\mu - \lambda}$$

# *Alternative Way for Sojourn Time Calculation*

- We know that $E[N] = \rho/(1-\rho)$
- We know arrival rate $\lambda$
- Then based on Little's Law
- $N = \lambda T$

$$\rightarrow \quad E[T]=E[N]/\lambda = 1/(\mu-\lambda)$$

# *M/M/1 Queue Example*

- A router's outgoing bandwidth is 100 kbps
- Arrival packet's number of bits has expo. distr. with mean number of 1 kbits
- Poisson arrival process: 80 packets/sec
- How many packets in router expected by a new arrival?
- What is the expected waiting time for a new arrival?
- What is the expected access delay (response time)?
- What is the prob. that the server is idle?
- What is P( N > 5 )?
- Suppose you can increase router bandwidth, what is the minimum bandwidth to support avg. access delay of 20ms?

# Sojourn Time Distribution

- T's pdf is denoted as $f_T(t)$, $t \geq 0$
- $T = X_1 + X_2 + \cdots + X_n + X$
  - Given there are N=n customers in the system
  - Then, T is sum of n+1 exponential distr.
    - T is (n+1)-order Erlang distr.
  - When conditioned on n, the pdf of T (n+1 order Erlang) is denoted as $f_{T|N}(t|n)$

$$f_{T|N}(t|n) = \frac{\mu(\mu t)^n e^{-\mu t}}{n!}$$

# Sojourn Time Distribution

- Remove condition N=n:
  - Remember P(N=n) = $\pi_n$ = (1-$\rho$)$\rho^n$

$$f_T(t) = f_{T|0}(t|0)\pi_0 + f_{T|1}(t|1)\pi_1 + \cdots$$

$$f_T(t) = \sum_{n=0}^{\infty}(1-\rho)\rho^n \frac{\mu(\mu t)^n e^{-\mu t}}{n!}$$

$$= (1-\rho)\mu e^{-\mu t}\sum_{n=0}^{\infty}(\rho\mu t)^n/n!$$

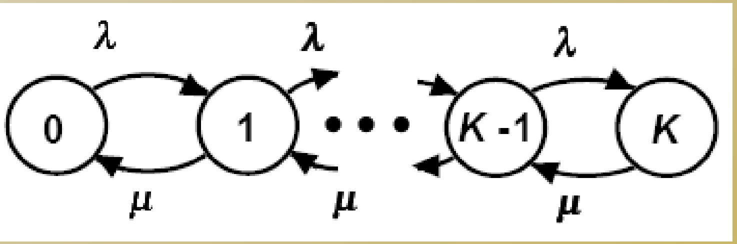
$$= (\mu-\lambda)e^{-\mu t}e^{\lambda t}$$

$$= (\mu-\lambda)e^{-(\mu-\lambda)t}$$

**Thus, T is exponential distr. with rate ($\mu$-$\lambda$)**

# M/M/1/K Queue

- Arrival: Poisson process with rate $\lambda$
- Service: exponential distr. with rate $\mu$
- Finite capacity of K customers
  - Customer arrives when queue is full is rejected
- Model as B-D process
  - N(t): no. of customers at time t
  - State transition diagram



# Calculation of $\pi_o$

- Balance equation:
  - $\pi_i = \rho\pi_{i-1} = \rho^i\pi_o$, i=1,⋯, K
- If $\lambda \neq \mu$:

$$\sum_{i=0}^{K} \pi_i = \pi_0 \sum_{i=0}^{K} \rho^i = \pi_0 \frac{1-\rho^{K+1}}{1-\rho}$$

$$\sum_{i=0}^{K} \pi_i = 1 \Rightarrow \quad \pi_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

- If $\lambda = \mu$: $\sum_{i=0}^{K} \pi_i = \pi_0 \sum_{i=0}^{K} \rho^i = (K+1)\pi_0$

$$\pi_i = 1/(K+1), i = 0, \cdots, K$$

# E[N]

- If $\lambda \neq \mu$:

$$E[N] = \sum_{i=0}^{K} i\pi_i$$

$$= \frac{1-\rho}{1-\rho^{K+1}} \sum_{i=0}^{K} \cdot i \cdot \rho^i$$

- If $\lambda = \mu$:

$$E[N] = \sum_{i=0}^{K} i\pi_i \quad = \frac{1}{K+1} \sum_{i=0}^{K} i$$

$$= \frac{1}{K+1} \frac{K(K+1)}{2} = \frac{K}{2}$$

# Throughput

- Throughput?
  - When not idle = $\mu$
  - When idle = 0
  - Throughput = $(1-\pi_o)\mu + \pi_o \cdot 0$
  - When not full = $\lambda$ (arrive pass)
  - When full = 0 (arrive drop)
    - Prob. Buffer overflow = $\pi_K$
  - Throughput = $(1-\pi_K)\lambda + \pi_K \cdot 0$

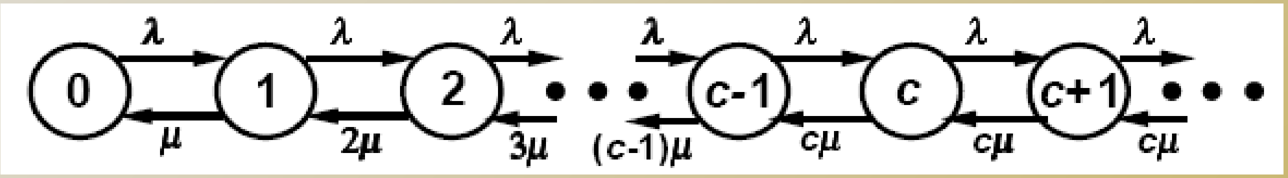
# Sojourn Time

- One way: $T = X_1+X_2+\cdots+X_n$ if there are n customers in ($n \leq K$)
  - Doable, but complicated
- Another way: Little's Law
  - $N = \lambda T$
  - The $\lambda$ means ***actual*** throughput

$$E[T] = \frac{E[N]}{\text{throughput}} = \frac{E[N]}{(1-\pi_0)\mu}$$

# M/M/c Queue

- c identical servers to provide service
- Model as B-D process, N(t): no.of customers
- State transition diagram:




- Balance equation:

$$\begin{cases} \lambda \pi_{i-1} & = i\mu\pi_i, \; i \leq c, \\ \lambda \pi_{i-1} & = c\mu\pi_i, \; i > c \end{cases}$$

- Solution to balance equation:

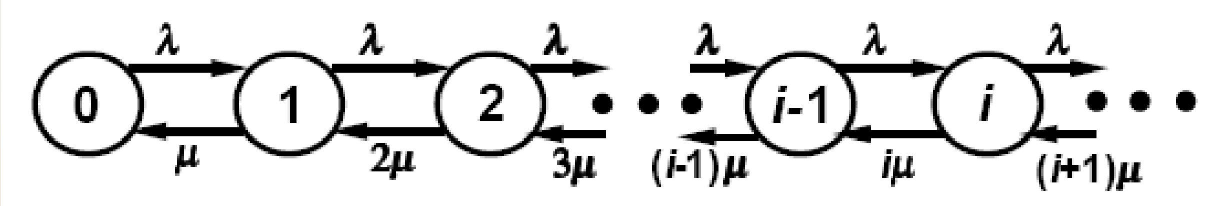
$$\pi_i = \begin{cases} \frac{\rho^i}{i!}\pi_0, & 0 \le i \le c, \\ \frac{\rho^i}{c!c^{i-c}}\pi_0, & c < i \end{cases}$$

- Prob. a customer has to wait (prob. of queuing)

$$P(queuing) = P(wait) = \sum_{n=c}^{\infty} \pi_n$$

# M/M/∞ Queue

- Infinite server (delay server)
  - Each user gets its own server for service
  - No waiting time




  - Balance equation:

$$\lambda \pi_{i-1} = i\mu\pi_i, \quad i = 0, 1, \cdots$$

$$\pi_i = \frac{\rho^i}{i!}\pi_0 = \frac{\rho^i}{i!}e^{-\rho}$$ **why?**

$$E[N] = \sum_{i=0}^{\infty} i\pi_i = \sum_{i=1}^{\infty} \frac{i\rho^i e^{-\rho}}{i!}$$

$$= \rho e^{-\rho} \sum_{i=1}^{\infty} \frac{\rho^{i-1}}{(i-1)!} = \rho$$

$$E[T] = \frac{E[N]}{\lambda} = \frac{1}{\mu}$$

Why?

# *PASTA property*

- PASTA: Poisson Arrivals See Time Average
- Meaning: When a customer arrives, it finds the same situation in the queueing system as an outside observer looking at the system at an arbitrary point in time.
- N(t): system state at time t
- Poisson arrival process with rate $\lambda$
- M(t): system at time t given that an arrival occurs in the next moment in $(t, t+\Delta t)$

$$P(M(t)=n) = P(N(t)=n|\text{arrival in}(t,t+\Delta t))$$

$$= \frac{P(N(t)=n, \text{arrival in } (t,t+\Delta t))}{P(\text{arrival in } (t,t+\Delta t))}$$

$$= \frac{P(N(t)=n)P(\text{arrival in } (t,t+\Delta t))}{P(\text{arrival in } (t,t+\Delta t))}$$

$$= P(N(t)=n)$$

- If not Poisson arrival, then not correct