UCF Stands For Opportunity

CDA6530: Performance Models of Computers and Networks

Chapter 5: Generating Random Number and Random Variables

SCHOOL OF ELECTRICAL ENGINEERING & COMPUTER SCIENCE

Objective

- Use computers to simulate stochastic processes
- Learn how to generate random variables
  - Discrete r.v.
  - Continuous r.v.
- Basis for many system simulations

Pseudo Random Number Generation (PRNG)

- $x_n = a\, x_{n-1} \mod m$
  - Multiplicative congruential generator
  - $x_n = \{0, 1, \cdots, m-1\}$
  - x_n/m is used to approx. distr. U(0,1)
  - x_0 is the initial “seed”
- Requirements:
  - No. of variables that can be generated before repetition begins is large
  - For any seed, the resultant sequence has the “appearance” of being independent
  - The values can be computed efficiently on a computer

- $x_n = a\, x_{n-1} \mod m$
- m should be a large prime number
- For a 32-bit machine (1 bit is sign)
  - $m=2^{31}-1 = 2,147,483,647$
  - $a = 7^5 = 16,807$
- For a 36-bit machine
  - $m= 2^{35}-31$
  - $a = 5^5$
- $x_n = (ax_{n-1} + c) \mod m$
  - Mixed congruential generator

In C Programming Language

- Int rand(void)
  - Return int value between 0 and RAND_MAX
  - RAND_MAX default value may vary between implementations but it is granted to be at least 32767
- X=rand()
  - X={0,1,⋯, RAND_MAX}
- X = rand()%m + n
  - X={n, n+1, ⋯, m+n-1}
  - Suitable for small m;
  - Lower numbers are more likely picked

(0,1) Uniform Distribution

- U(0,1) is the basis for random variable generation
- C code (at least what I use):

```
Double rand01(){
    double temp;
    temp = double( rand()+0.5 ) /
    (double(RAND_MAX) + 1.0);
    return temp;
}
```

Generate Discrete Random Variables ---- Inverse Transform Method

- r.v. X: $P(X = x_j) = p_j, \quad j=0,1,\cdots$
- We generate a PRNG value $U \sim U(0,1)$
  - For $0<a<b<1$, $P(a \leq U <b\} = b-a$, thus

$$P(X = x_j) = P(\sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i) = p_j$$

$$X = \begin{cases} x_0 & \text{if } U < p_0 \\ x_1 & \text{if } p_0 \leq U < p_0 + p_1 \\ \vdots & \\ x_j & \text{if } \sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^{j} p_i \\ \vdots & \end{cases}$$

Example

- A loaded dice:
  - P(1)=0.1; P(2)=0.1; P(3)=0.15; P(4)=0.15
  - P(5)=0.2; P(6)=0.3
- Generate 1000 samples of the above loaded dice throwing results
  - How to write the Matlab code?

Generate a Poisson Random Variable

$$p_i = P(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}, \quad i = 0, 1, \cdots$$

- Use following recursive formula to save computation:

$$p_{i+1} = \frac{\lambda}{i+1} p_i$$

Some Other Approaches

- Acceptance-Rejection approach
- Composition approach

  - They all assume we have already generated a random variable first (not U)
  - Not very useful considering our simulation purpose

Generate Continuous Random Variables ---- Inverse Transform Method

- r.v. X: $F(x) = P(X \le x)$
- r.v. Y: $Y = F^{-1}(U)$
  - Y has distribution of F. ($Y =_{st} X$)
- $P(Y \le x) = P(F^{-1}(U) \le x)$

$$= P(F(F^{-1}(U)) \le F(x))$$
$$= P(U \le F(x))$$
$$= P(X \le x)$$

- Why? Because $0<F(x)<1$ and the CDF of a uniform $F_U(y) = y$ for all $y \in [0; 1]$

Generate Exponential Random Variable

$$F(x) = 1 - e^{-\lambda x}$$

$$U = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - U$$

$$x = -\ln(1 - U)/\lambda$$

$$F^{-1}(U) = -\ln(1 - U)/\lambda$$

Generate Normal Random Variable --- Polar method

- The theory is complicated, we only list the algorithm here:
  - Objective: Generate a pair of independent standard normal r.v. ~ N(0, 1)
    - Step 1: Generate (0,1) random number U_1 and U_2
    - Step 2: Set $V_1 = 2U_1 – 1$, $V_2 = 2U_2-1$ $\quad S = V_1^2 + V_2^2$
    - Step 3: If S> 1, return to Step 1.
    - Step 4: Return two standard normal r.v.:

$$X = \sqrt{\frac{-2 \ln S}{S}} V_1, \quad Y = \sqrt{\frac{-2 \ln S}{S}} V_2$$

z	F(x)	z	F(x)	z	F(x)
-2.5	0.006	-1	0.159	0.5	0.691
-2.4	0.008	-0.9	0.184	0.6	0.726
-2.3	0.011	-0.8	0.212	0.7	0.758
-2.2	0.014	-0.7	0.242	0.8	0.788
-2.1	0.018	-0.6	0.274	0.9	0.816
-2	0.023	-0.5	0.309	1	0.841
-1.9	0.029	-0.4	0.345	1.1	0.864
-1.8	0.036	-0.3	0.382	1.2	0.885
-1.7	0.045	-0.2	0.421	1.3	0.903
-1.6	0.055	-0.1	0.46	1.4	0.919
-1.5	0.067	0	0.5	1.5	0.933
-1.4	0.081	0.1	0.54	1.6	0.945
-1.3	0.097	0.2	0.579	1.7	0.955
-1.2	0.115	0.3	0.618	1.8	0.964
-1.1	0.136	0.4	0.655	1.9	0.971

- Another approximate method- Table lookup
  - Treat Normal distr. r.v. X as discrete r.v.
  - Generate a U, check U with F(x) in table, get z

Generate Normal Random Variable

- Polar method generates a pair of standard normal r.v.s X~N(0,1)
- What about generating r.v. Y~N(μ, σ^2)?

  - Y= σX + μ

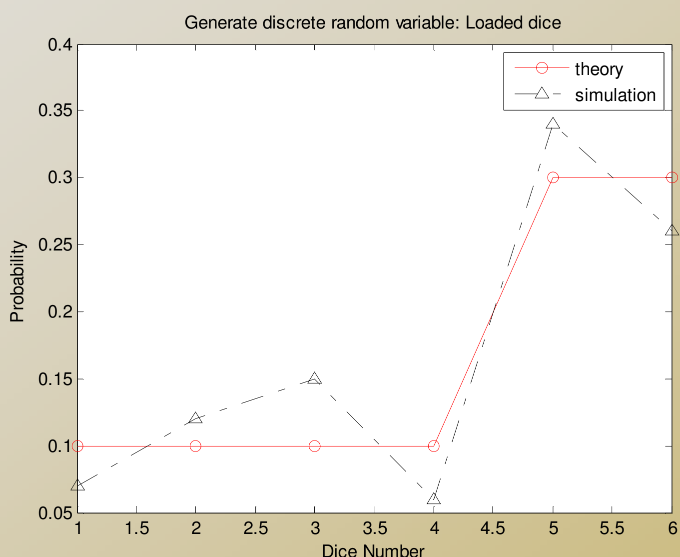
Generating a Random Permutation

- Generate a permutation of $\{1,\cdots, n\}$
- $Int(kU) + 1$:
  - uniformly pick from $\{1,2,\cdots, k\}$
- Algorithm:
  - $P_1, P_2, \cdots, P_n$ is a permutation of $1,2,\cdots, n$ (e.g., we can let $P_j = j$, $j=1,\cdots, n$)
  - Set $k = n$
  - Generate U, let $I = Int(kU)+1$
  - Interchange the value of P_I and P_k
  - Let $k=k-1$ and if $k>1$ goto Step 3
  - $P_1, P_2, \cdots, P_n$ is a generated random permutation

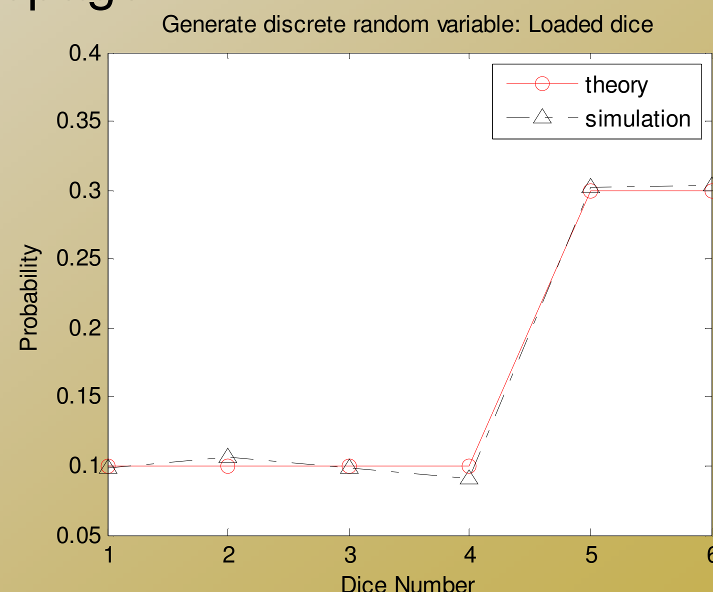
Example: permute (10, 20, 30, 40, 50)

Example of Generating Random Variable

- A loaded dice has the pmf:
  - P(X=1)=P(2)=P(3) =P(4)= 0.1, P(5)=P(6) = 0.3
- Generate 100 samples, compare pmf of simulation with pmf of theoretical values
  - Matlab code is on course webpage



100 samples



1000 samples

Monte Carlo Approach ---- Use Random Number to Evaluate Integral

$$\theta = \int_0^1 g(x)dx \qquad \theta = E[g(U)]$$

- U is uniform distr. r.v. (0,1)
- Why?

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

$$f_U(x) = 1 \quad \text{if } 0 < x < 1$$

- $U_1, U_2, \cdots, U_k$ are independent generated uniform distr. (0,1)
  - $g(U_1), \cdots, g(U_k)$ are independent
  - Law of large number:

$$\sum_{i=1}^{k} \frac{g(U_i)}{k} \to E[g(U)] = \theta \text{ as } k \to \infty$$

$$\theta = \int_a^b g(x)dx$$

- Substitution: y=(x-a)/(b-a), dy = dx/(b-a)

$$\theta = \int_0^1 (b-a) \cdot g(a + (b-a)y)dy = \int_0^1 h(y)dy$$

$$h(y) = (b-a) \cdot g(a + (b-a)y)$$

$$\theta = \int_0^1 \int_0^1 \cdots \int_0^1 g(x_1, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

$$\theta = E[g(U_1, \cdots, U_n)]$$

- Generate many g(….)
- Compute average value
    - which is equal to θ