

UCF



Stands For Opportunity

CDA6530: Performance Models of Computers and Networks

***Chapter 5: Generating Random Number
and Random Variables***

Objective

- ❑ Use computers to simulate stochastic processes
- ❑ Learn how to generate random variables
 - ❑ Discrete r.v.
 - ❑ Continuous r.v.
- ❑ Basis for many system simulations

Pseudo Random Number Generation (PRNG)

- ❑ $x_n = a x_{n-1} \bmod m$
 - ❑ Multiplicative congruential generator
 - ❑ $x_n = \{0, 1, \dots, m-1\}$
 - ❑ x_n/m is used to approx. distr. $U(0,1)$
 - ❑ x_0 is the initial “seed”
- ❑ **Requirements:**
 - ❑ No. of variables that can be generated before repetition begins is large
 - ❑ For any seed, the resultant sequence has the “appearance” of being independent
 - ❑ The values can be computed efficiently on a computer

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- ❑ $x_n = a x_{n-1} \text{ mod } m$
 - ❑ m should be a large prime number
 - ❑ For a 32-bit machine (1 bit is sign)
 - ❑ $m = 2^{31} - 1 = 2,147,483,647$
 - ❑ $a = 7^5 = 16,807$
 - ❑ For a 36-bit machine
 - ❑ $m = 2^{35} - 31$
 - ❑ $a = 5^5$
 - ❑ $x_n = (ax_{n-1} + c) \text{ mod } m$
 - ❑ Mixed congruential generator

In C Programming Language

- ❑ **Int rand(void)**
 - ❑ Return int value between 0 and RAND_MAX
 - ❑ RAND_MAX default value may vary between implementations but it is granted to be at least 32767
- ❑ **X=rand()**
 - ❑ $X=\{0, 1, \dots, \text{RAND_MAX}\}$
- ❑ **$X = \text{rand()} \% m + n$**
 - ❑ $X=\{n, n+1, \dots, m+n-1\}$
 - ❑ Suitable for small m;
 - ❑ Lower numbers are more likely picked

(0,1) Uniform Distribution

- ❑ U(0,1) is the basis for random variable generation
- ❑ C code (at least what I use):

```
Double rand01(){  
    double temp;  
    temp = double( rand()+0.5 ) /  
    (double(RAND_MAX) + 1.0);  
    return temp;  
}
```

Generate Discrete Random Variables

---- Inverse Transform Method

- r.v. X : $P(X = x_j) = p_j, \quad j=0,1,\dots$
- We generate a PRNG value $U \sim U(0,1)$
 - For $0 < a < b < 1$, $P(a \leq U < b) = b - a$, thus

$$P(X = x_j) = P\left(\sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^j p_i\right) = p_j$$

$$X = \begin{cases} x_0 & \text{if } U < p_0 \\ x_1 & \text{if } p_0 \leq U < p_0 + p_1 \\ \vdots & \\ x_j & \text{if } \sum_{i=0}^{j-1} p_i \leq U < \sum_{i=0}^j p_i \\ \vdots & \end{cases}$$

Example

- **A loaded dice:**
 - $P(1)=0.1$; $P(2)=0.1$; $P(3)=0.15$; $P(4)=0.15$
 - $P(5)=0.2$; $P(6)=0.3$
- **Generate 1000 samples of the above loaded dice throwing results**
 - How to write the Matlab code?

Generate a Poisson Random Variable

$$p_i = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}, \quad i = 0, 1, \dots$$

- Use following recursive formula to save computation:

$$p_{i+1} = \frac{\lambda}{i+1} p_i$$

Some Other Approaches

- ❑ Acceptance-Rejection approach
- ❑ Composition approach
- ❑ They all assume we have already generated a random variable first (not U)
- ❑ Not very useful considering our simulation purpose

Generate Continuous Random Variables

---- Inverse Transform Method

- r.v. X : $F(x) = P(X \leq x)$
- r.v. Y : $Y = F^{-1}(U)$
 - Y has distribution of F . ($Y \stackrel{\text{st}}{=} X$)
- $P(Y \leq x) = P(F^{-1}(U) \leq x)$
 - $= P(F(F^{-1}(U)) \leq F(x))$
 - $= P(U \leq F(x))$
 - $= P(X \leq x)$
- Why? Because $0 < F(x) < 1$ and the CDF of a uniform $F_U(y) = y$ for all $y \in [0; 1]$

Generate Exponential Random Variable

$$F(x) = 1 - e^{-\lambda x}$$

$$U = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - U$$

$$x = -\ln(1 - U)/\lambda$$

$$F^{-1}(U) = -\ln(1 - U)/\lambda$$

Generate Normal Random Variable --- Polar method

- The theory is complicated, we only list the algorithm here:
 - Objective: Generate a pair of independent standard normal r.v. $\sim N(0, 1)$
 - Step 1: Generate (0,1) random number U_1 and U_2
 - Step 2: Set $V_1 = 2U_1 - 1$, $V_2 = 2U_2 - 1$ $S = V_1^2 + V_2^2$
 - Step 3: If $S > 1$, return to Step 1.
 - Step 4: Return two standard normal r.v.:

$$X = \sqrt{\frac{-2 \ln S}{S}} V_1, \quad Y = \sqrt{\frac{-2 \ln S}{S}} V_2$$

z	F(x)	z	F(x)	z	F(x)
-2.5	0.006	-1	0.159	0.5	0.691
-2.4	0.008	-0.9	0.184	0.6	0.726
-2.3	0.011	-0.8	0.212	0.7	0.758
-2.2	0.014	-0.7	0.242	0.8	0.788
-2.1	0.018	-0.6	0.274	0.9	0.816
-2	0.023	-0.5	0.309	1	0.841
-1.9	0.029	-0.4	0.345	1.1	0.864
-1.8	0.036	-0.3	0.382	1.2	0.885
-1.7	0.045	-0.2	0.421	1.3	0.903
-1.6	0.055	-0.1	0.46	1.4	0.919
-1.5	0.067	0	0.5	1.5	0.933
-1.4	0.081	0.1	0.54	1.6	0.945
-1.3	0.097	0.2	0.579	1.7	0.955
-1.2	0.115	0.3	0.618	1.8	0.964
-1.1	0.136	0.4	0.655	1.9	0.971

- **Another approximate method- Table lookup**
 - Treat Normal distr. r.v. X as discrete r.v.
 - Generate a U , check U with $F(x)$ in table, get z

Generate Normal Random Variable

- ❑ Polar method generates a pair of standard normal r.v.s $X \sim N(0,1)$
- ❑ What about generating r.v. $Y \sim N(\mu, \sigma^2)$?
 - ❑ $Y = \sigma X + \mu$

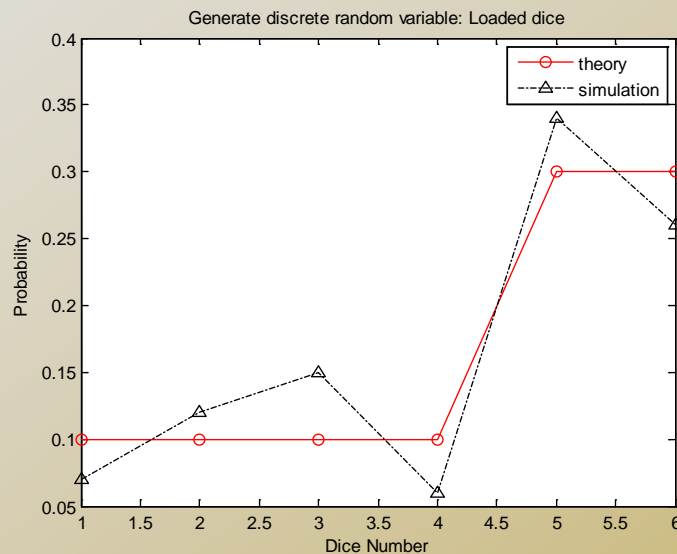
Generating a Random Permutation

- Generate a permutation of $\{1, \dots, n\}$
- $\text{Int}(kU) + 1$:
 - uniformly pick from $\{1, 2, \dots, k\}$
- **Algorithm:**
 - P_1, P_2, \dots, P_n is a permutation of $1, 2, \dots, n$ (e.g., we can let $P_j = j, j = 1, \dots, n$)
 - Set $k = n$
 - Generate U , let $I = \text{Int}(kU) + 1$
 - Interchange the value of P_I and P_k
 - Let $k = k - 1$ and if $k > 1$ goto Step 3
 - P_1, P_2, \dots, P_n is a generated random permutation

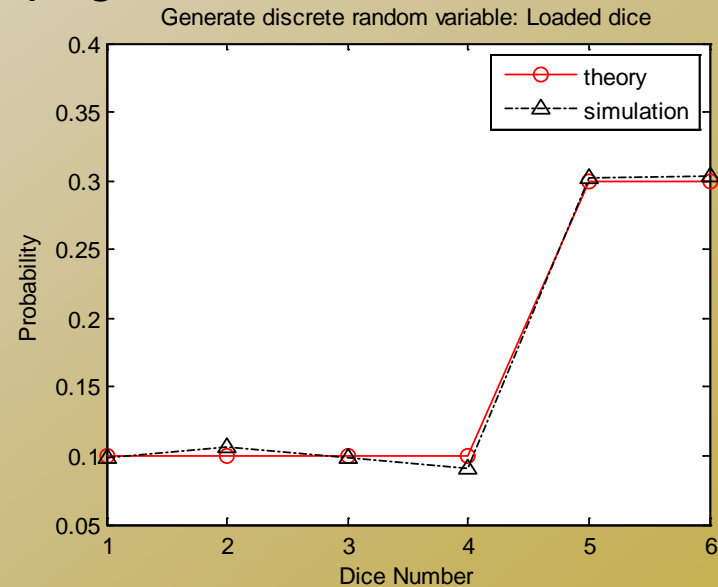
Example: permute (10, 20, 30, 40, 50)

Example of Generating Random Variable

- A loaded dice has the pmf:
 - $P(X=1)=P(2)=P(3)=P(4)=0.1$, $P(5)=P(6)=0.3$
- Generate 100 samples, compare pmf of simulation with pmf of theoretical values
 - Matlab code is on course webpage



100 samples



1000 samples

Monte Carlo Approach ----

Use Random Number to Evaluate Integral

$$\theta = \int_0^1 g(x) dx \qquad \theta = E[g(U)]$$

- **U is uniform distr. r.v. (0,1)**
- **Why?**

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx$$

$$f_U(x) = 1 \quad \text{if } 0 < x < 1$$

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- U_1, U_2, \dots, U_k are independent generated uniform distr. (0,1)
 - $g(U_1), \dots, g(U_k)$ are independent
 - Law of large number:

$$\sum_{i=1}^k \frac{g(U_i)}{k} \rightarrow E[g(U)] = \theta \text{ as } k \rightarrow \infty$$

$$\theta = \int_a^b g(x) dx$$

- **Substitution: $y=(x-a)/(b-a)$, $dy = dx/(b-a)$**

$$\theta = \int_0^1 (b - a) \cdot g(a + (b - a)y) dy = \int_0^1 h(y) dy$$

$$h(y) = (b - a) \cdot g(a + (b - a)y)$$

$$\theta = \int_0^1 \int_0^1 \cdots \int_0^1 g(x_1, \cdots, x_n) dx_1 dx_2 \cdots dx_n$$

$$\theta = E[g(U_1, \cdots, U_n)]$$

- Generate many $g(\dots)$
- Compute average value
 - which is equal to θ