UCF Stands For Opportunity

CDA6530: Performance Models of Computers and Networks

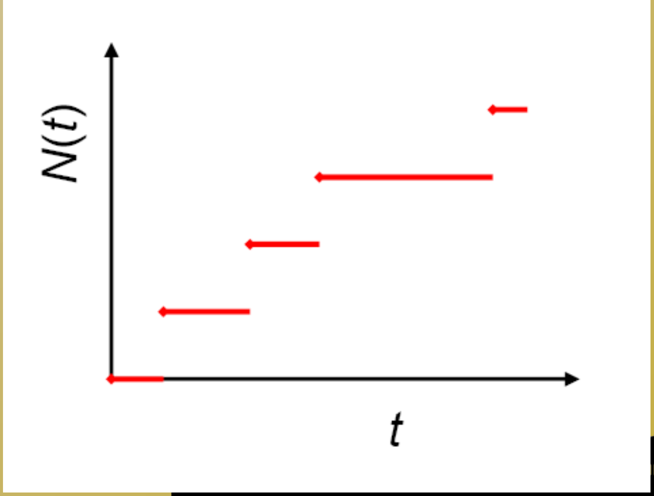
Chapter 3: Review of Practical Stochastic Processes

Definition

- Stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables (rvs); one rv for each $X(t)$ for each $t \in T$
  - Index set T --- set of possible values of t
    - t only means time
    - T: countable → discrete-time process
    - T: real number → continuous-time process
  - State space --- set of possible values of $X(t)$

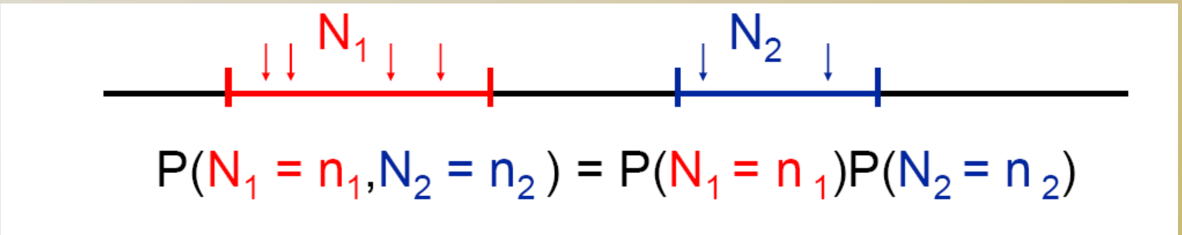
Counting Process

- A stochastic process that represents no. of events that occurred by time t; a continuous-time, discrete-state process {N(t), t>0} if
  - N(0)=0
  - $N(t) \geq 0$
  - N(t) increasing (non-decreasing) in t
  - N(t)-N(s) is the Number of events happen in time interval [s, t]



Counting Process

- Counting process has independent increments if no. events in disjoint intervals are independent
  - $P(N_1=n_1, N_2=n_2) = P(N_1=n_1)P(N_2=n_2)$ if N_1 and N_2 are disjoint intervals



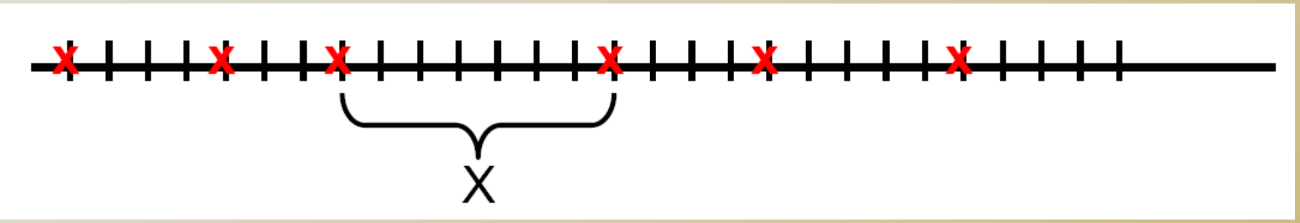
- counting process has stationary increments if no. of events in $[t_1+s; t_2+s]$ has the same distribution as no. of events in $[t_1; t_2]$; $s > 0$

Bernoulli Process

- N_t: no. of successes by time t=0,1,…is a counting process with independent and stationary increments
  - p: prob. of success
  - Note: t is discrete time
- When $n \leq t$, $$P(N_t = n) = \binom{t}{n} p^n (1-p)^{t-n}$$
  - $N_t \sim B(t, p)$
- $E[N_t]=tp$, $Var[N_t]=tp(1-p)$

Bernoulli Process

- X: time between success
  - Geometric distribution
  - $P(X=n) = (1-p)^{n-1}p$



Little o notation

- Definition: f(h) is o(h) if

$$\lim_{h \to 0} \frac{f(h)}{h} = 0$$

  - $f(h)=h^2$ is o(h)
  - f(h)=h is not
  - $f(h)=h^r$, r>1 is o(h)
  - sin(h) is not
  - If f(h) and g(h) are o(h), then f(h)+g(h)=o(h)

  - Note: h is continuous

Example: Exponential R.V.

- Exponential r.v. X with parameter λ has PDF $P(X<h) = 1-e^{-\lambda h}$, h>0

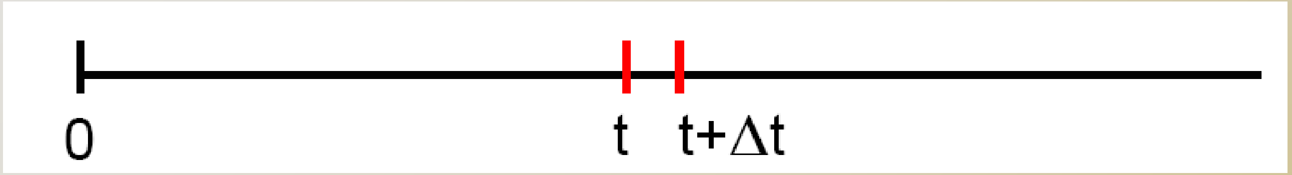
$$
\begin{aligned}
P[X \le t+h | X > t] &= P[X \le h] && \text{Why?} \\
&= 1 - e^{-\lambda h} \\
&= 1 - [1 - \lambda h + \sum_{n=2}^{\infty} \frac{(\lambda h)^n}{n!}] && \text{Why?} \\
&= \lambda h + o(h)
\end{aligned}
$$

Poisson Process

- Counting process $\{N(t), t \geq 0\}$ with rate λ
  - t is continuous
  - $N(0)=0$
  - Independent and stationary increments
  - $P(N(h)=1) = \lambda h + o(h)$
  - $P(N(h) \geq 2) = o(h)$
  - Thus, $P(N(h)=0) = ?$
    - $P(N(h)=0) = 1 - \lambda h + o(h)$

- Notation: $P_n(t) = P(N(t)=n)$

Drift Equations



$$\begin{aligned}
P_n(t+\Delta t) &= P_{n-1}(t)\lambda\Delta t + P_n(t)(1-\lambda\Delta t) + o(\Delta t)\\
P_n(t+\Delta t) - P_n(t) &= P_{n-1}(t)\lambda\Delta t - P_n(t)\lambda\Delta t + o(\Delta t)\\
\frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} &= P_{n-1}(t)\lambda - P_n(t)\lambda + \frac{o(\Delta t)}{\Delta t}
\end{aligned}$$

Taking limit $\Delta t \to 0$, we get

$$\frac{dP_n}{dt} = \lambda P_{n-1} - \lambda P_n$$

- For n=0, $P_0(t+\Delta t) = P_0(t)(1-\lambda\Delta t)+o(\Delta t)$
  - Thus, $dP_0(t)/dt = -\lambda P_0(t)$
  - Thus, $P_0(t) = e^{-\lambda t}$ Why?
  - Thus, inter-arrival time is exponential distr. With the same rate λ
    - Remember exponential r.v.: $F_X(x)= 1-e^{-\lambda x}$
    - That means: $P(X> t) = e^{-\lambda t}$
    - $\{X>t\}$ means at time t, there is still no arrival
  - $X^{(n)}$: time for n consecutive arrivals
    - Erlang r.v. with order k

$$f(x;k,\lambda) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!} \quad \text{for } x > 0.$$

$$\frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t)$$

$$P_0(t) = e^{-\lambda t}$$

$$\text{Solution: } P_n(t) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

- Similar to Poisson r.v. $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$
  - You can think Poisson r.v. is the static distr. of a Poisson process at time t

Poisson Process

- Take i.i.d. sequence of exponential rvs $\{X_i\}$ with rate λ
  - Define: $N(t) = \max\{n \mid \Sigma_{1 \leq i \leq n} X_i \leq t\}$,
  - $\{N(t)\}$ is a Poisson process
  - Meaning: Poisson process is composed of many independent arrivals with exponential inter-arrival time.

Poisson Process

- if *N*(*t*) is a Poisson process and one event occurs in [0,*t*], then the time to the event, denoted as r.v. X, is uniformly distributed in [0,*t*],
  - $f_{X|N(t)=1}(x|1)=1/t, \quad 0\leq x\leq t$

- Meaning:
  - Given an arrival happens, it could happen at any time
  - Exponential distr. is memoryless
  - One reason why call the arrival with "rate" λ
    - Arrival with the same prob. at any time

Poisson Process

- if $N_1(t)$ and $N_2(t)$ are ***independent*** Poisson processes with rates λ_1 and λ_2, then $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$

- Intuitive explanation:
  - A Poisson process is caused by many independent entities (n) with small chance (p) arrivals
    - Arrival rate is proportional to population size λ=np
  - Still a Poisson proc. if two large groups of entities arrives in mixed format

Poisson Process

- $N(t)$ is Poisson proc. with rate λ , M_i is Bernoulli proc. with success prob. p. Construct a new process $L(t)$ by only counting the n-th event in $N(t)$ whenever $M_n > M_{n-1}$ (i.e., success at time n)
- L(t) is Poisson with rate λp
  - Useful in analysis based on random sampling

Example 1

- A web server where failures are described by a Poisson process with rate $\lambda = 2.4$/day, i.e., the time between failures, *X*, is exponential r.v. with mean E[*X*] = 10hrs.
  - P(time between failures < *1* day) =
  - P(5 failures in 1 day)=
  - P(N(5)<10)=
  - look in on system at random day, what is prob. of no. failures during next 24 hours?
  - failure is memory failure with prob. 1/9, CPU failure with prob. 8/9. Failures occur as independent events. What is process governing memory failures?

Example 2

The arrival of claims at an insurance company follows a Poisson process. On average the company gets 100 claims per week. Each claim follows an exponential distribution with mean $700.00. The company offers two types of policies. The first type has no deductible and the second has a $250.00 deductible. If the claim sizes and policy types are independent of each other and of the number of claims, and twice as many policy holders have deductibles as not, what is the mean liability amount of the company in any 13 week period?

- First, claims be split into two Poisson arrival processes
  - X: no deductible claims   Y: deductible claims
- Second, the formula for liability?

Birth-Death Process

- Continuous-time, discrete-space stochastic process $\{N(t), t > 0\}$, $N(t) \in \{0, 1, ...\}$
- N(t): population at time t
  - $P(N(t+h) = n+1 \mid N(t) = n) = \lambda_n h + o(h)$
  - $P(N(t+h) = n-1 \mid N(t) = n) = \mu_n h + o(h)$
  - $P(N(t+h) = n \mid N(t) = n) = 1-(\lambda_n + \mu_n) h + o(h)$
  - λ_n - birth rates
  - μ_n - death rates, $\mu_0 = 0$

- Q: what is $P_n(t) = P(N(t) = n)$? $n = 0,1,...$

Birth-Death Process

- Similar to Poisson process drift equation

$$dP_n(t)/dt = P_{n-1}(t)\,\lambda_{n-1} + P_{n+1}(t)\,\mu_{n+1} - (\lambda_n + \mu_n)\,P_n(t), \qquad n=1,\ldots$$

$$dP_0(t)/dt = P_1(t)\,\mu_1 - \lambda_0\,P_0(t)$$

Initial condition: $P_n(0)$

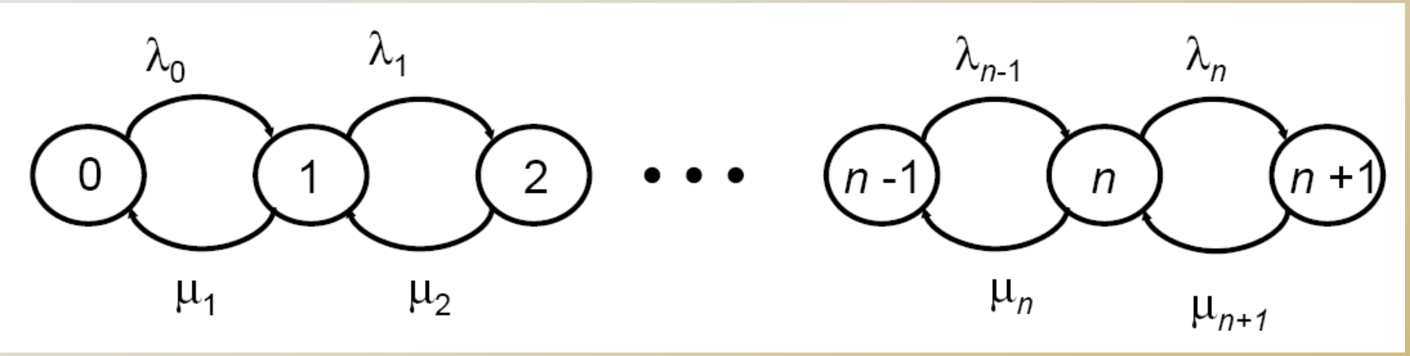
- If $\mu_i=0$, $\lambda_i=\lambda$, then B-D process is a Poisson process

Stationary Behavior of B-D Process

- Most real systems reach equilibrium as $t \rightarrow \infty$
  - No change in $P_n(t)$ as t changes
  - No dependence on initial condition
- $P_n = \lim_{t \rightarrow \infty} P_n(t)$
- Drift equation becomes:

$$(\lambda_n + \mu_n)\, P_n = \lambda_{n-1}\, P_{n-1} + \mu_{n+1}\, P_{n+1}$$

Transition State Diagram



- Balance Equations:
  - Rate of trans. into n = rate of trans. out of n

$$\lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1} = (\lambda_n + \mu_n) P_n, \qquad n \geq 1$$

$$\mu_1 P_1 = \lambda_0 P_0,$$

  - Rate of trans. to left = rate of trans. to right

$$\lambda_{n-1} P_{n-1} = \mu_n P_n$$

- Probability requirement:

$$\sum_{n=0}^{\infty} P_n = 1$$

Markov Process

- Prob. of future state depends only on present state
- $\{X(t), t>0\}$ is a MP if for any set of time $t_1<\cdots<t_{n+1}$ and any set of states $x_1<\cdots<x_{n+1}$
  - $P(X(t_{n+1})=x_{n+1}|X(t_1)=x_1, \cdots X(t_n)=x_n\}$
    $= P(X(t_{n+1})=x_{n+1}| X(t_n)=x_n\}$
- B-D process, Poisson process are MP

Markov Chain

- Discrete-state MP is called Markov Chain (MC)
  - Discrete-time MC
  - Continuous-time MC
- First, consider discrete-time MC

$$P_{ij} = P(X_{n+1} = j \mid X_n = i), i, j = 0, 1, \ldots ; n \geq 0$$

- Define transition prob. matrix:

$$\mathbf{P} = [P_{ij}]$$

Chapman-Kolmogorov Equation

- What is the state after n transitions?
- A: define $P_{ij}^n = P(X_{n+m} = j \mid X_m = i),\ n \geq 0,\ i,j \geq 0$

$$
\begin{aligned}
P_{ij}^{n+m} &= P(X_{n+m} = j \mid X_0 = i), \\
&= \sum_{k=0}^{\infty} P(X_{n+m} = j, X_n = k \mid X_0 = i), \quad \text{Why?} \\
&= \sum_{k=0}^{\infty} P(X_{n+m} = j \mid X_n = k, X_0 = i) P(X_n = k \mid X_0 = i), \\
&= \sum_{k=0}^{\infty} P(X_{n+m} = j \mid X_n = k) P(X_n = k \mid X_0 = i), \quad \text{Why?} \\
&= \sum_{k=0}^{\infty} P_{kj}^m P_{ik}^n
\end{aligned}
$$

- If MC has n state

$$P_{ij}^2 = \sum_{k=1}^{n} P_{ik}P_{kj} \qquad \Rightarrow [P_{ij}^2] = \mathbf{P}\cdot\mathbf{P}$$

- Define n-step transition prob. matrix:

$$\mathbf{P}^{(n)} = [P_{ij}^n]$$

- C-K equation means:

$$\mathbf{P}^{(n+m)} = \mathbf{P}^{(n)} \cdot \mathbf{P}^{(m)}$$

$$\mathbf{P}^{(n)} = \mathbf{P}\cdot\mathbf{P}^{(n-1)} = \mathbf{P}^n$$

Markov Chain

- Irreducible MC:
  - If every state can be reached from any other states
- Periodic MC:
  - A state i has period k if any returns to state i occurs in multiple of k steps
  - k=1, then the state is called aperiodic
  - MC is aperiodic if all states are aperiodic

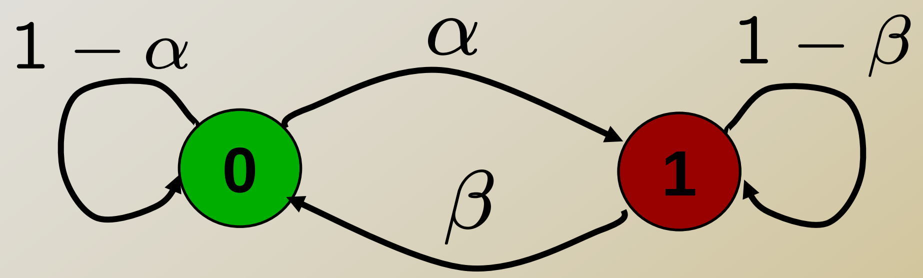
- An irreducible, aperiodic finite-state MC is ergodic, which has a stationary (steady-state) prob. distr.

$$\pi = (\pi_0, \pi_1, \cdots, \pi_n)$$

$$\begin{aligned} \pi &= \pi P, \\ \pi \mathbf{1} &= 1 \end{aligned}$$

where $\mathbf{1} = (1 \ \ldots)^T$

Example



- Markov on-off model (or 0-1 model)
- Q: the steady-state prob.?

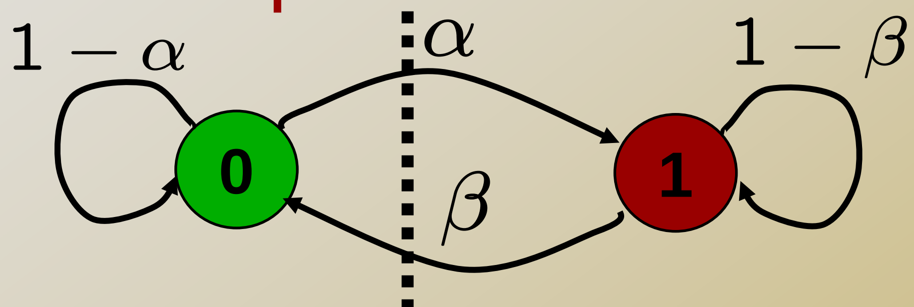
$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$$\begin{cases} \pi_0 = & (1-\alpha)\pi_0 + \beta\pi_1 \\ \pi_1 = & \alpha\pi_0 + (1-\beta)\pi_1 \end{cases} \Rightarrow \begin{cases} \pi_0 & = \frac{\beta}{\alpha+\beta} \\ \pi_1 & = \frac{\alpha}{\alpha+\beta} \end{cases}$$

$$\pi_0 + \pi_1 = 1$$

An Alternative Calculation

- Use balance equation:



- Rate of trans. to left = rate of trans. to right

$$\alpha\pi_0 = \beta\pi_1$$

$$\pi_0 + \pi_1 = 1$$

$$\Rightarrow \begin{cases} \pi_0 & = \frac{\beta}{\alpha+\beta} \\ \pi_1 & = \frac{\alpha}{\alpha+\beta} \end{cases}$$

Discrete-Time MC State Staying Time

- X_i: the number of time steps a MC stays in the same state i
- $P(X_i = k) = P_{ii}^{k-1} (1-P_{ii})$
  - X_i follows geometric distribution
  - Average time: $1/(1-P_{ii})$

- In continuous-time MC, the staying time is?
  - Exponential distribution time

Homogeneous Continuous-Time Markov Chain

- $P(X(t+h)=j|X(t)=i) = \lambda_{ij}h + o(h)$
- We have the properties:

$$P(X(t+h)=i|X(t)=i) = 1 - \sum_{j\neq i} \lambda_{ij}h + o(h)$$

$$P(X(t+h)\neq i|X(t)=i) = \sum_{j\neq i} \lambda_{ij}h + o(h)$$

  - The state holding time is exponential distr. with rate $\lambda = \sum_{j\neq i} \lambda_{ij}$
    - Why?
    - Due to the summation of independent exponential distr. is still exponential distr.

Steady-State

- Ergodic continuous-time MC
- Define π_i = P(X=i)
- Consider the state transition diagram
  - Transit out of state i = transit into state i

$$\pi_i \sum_{j \neq i} \lambda_{ij} = \sum_{j \neq i} \pi_j \lambda_{ji}$$

$$\sum_i \pi_i = 1$$

Infinitesimal Generator

- Define Q = [q_{ij}] where

$$q_{ij} = \begin{cases} -\sum_{k \neq i} \lambda_{ik} & \text{when } i = j \\ \lambda_{ij} & i \neq j \end{cases}$$

- Q is called infinitesimal generator

$$\pi = [\pi_1 \; \pi_2 \; \cdots]$$

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

Why?

Discrete vs. Continues MC

- Discrete
  - Jump at time tick
  - Staying time: geometric distr.
  - Transition matrix P
  - Steady state:

$$\begin{aligned} \pi &= \pi P, \\ \pi \mathbf{1} &= 1 \end{aligned}$$

  - State transition diagram:
    - Has self-jump loop
    - Probability on arc
- Continuous
  - Jump at continuous t
  - Staying time: exponential distr.
  - Infinitesimal generator Q
  - Steady state:

$$\pi \mathbf{Q} = 0$$

$$\pi \mathbf{1} = 1$$

  - State transition diagram:
    - No self-jump loop
    - Transition rate on arc

Semi-Markov Process

- X(t): discrete state, continuous time
  - State jump: follow Markov Chain Z_n
  - State i holding time: follow a distr. $Y^{(i)}$
- If $Y^{(i)}$ follows exponential distr. λ
  - X(t) is a continuous-time Markov Chain

Steady State

- Let $\pi'_j = \lim_{n\to\infty} P(Z_n=j)$
- Let $\pi_j = \lim_{t\to\infty} P(X(t)=j)$

$$\pi_j = \frac{\pi'_j E[Y^{(j)}]}{\sum_{i\in S} \pi'_i E[Y^{(i)}]}, \quad j \in S$$

Why?