

CDA6530: Performance Models of Computers and Networks

Review of Transform Theory

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- **Why using transform?**
 - Make analysis easier
 - **Two transforms for probability**
 - Non-negative integer r.v.
 - Z-transform (or called probability generating function (pgf))
 - Non-negative, real valued r.v.
 - Laplace transform (LT)

Z-transform

- **Definition:**

- $G_X(z)$ is Z-transform for r.v. X

$$G_X(z) \equiv E[z^X] = \sum_{k=0}^{\infty} p_k z^k$$

- **Example:**

- X is geometric r.v., $p_k = (1-p)p^k$

$$G_X(z) = \sum_{k=0}^{\infty} (1-p)p^k z^k = \frac{1-p}{1-pz},$$

For $pz < 1$

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- Poisson distr., $p_k = \lambda^k e^{-\lambda}/k!$

$$G_X(z) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} z^k$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} (\lambda z)^k / k!$$

$$= e^{-\lambda(1-z)}$$

Benefit

$$\frac{dG_X(z)}{dz} = \sum_{k=1}^{\infty} k p_k z^{k-1}$$

□ Thus

$$E[X] = \sum_{k=1}^{\infty} k p_k z^{k-1} \Big|_{z=1} = \frac{dG_X(z)}{dz} \Big|_{z=1}$$

$$\frac{d^2 G_X(z)}{dz^2} \Big|_{z=1} = E[X^2] - E[X]$$

□ Convolution: X, Y independent with pdfs $G_X(z)$ and $G_Y(z)$, Let $U=X+Y$

$$G_U(z) = G_X(z)G_Y(z)$$

Solution of M/M/1 Using Transform

$$\begin{aligned}(\lambda + \mu)\pi_i &= \lambda\pi_{i-1} + \mu\pi_{i+1}, \quad i = 1, \dots \\ \lambda\pi_0 &= \mu\pi_1\end{aligned}$$

- Multiplying by z^i , using $\rho = \lambda/\mu$, and summing over i

$$(1 + \rho) \sum_{i=0}^{\infty} \pi_i z^i = \rho z \sum_{i=0}^{\infty} \pi_i z^i + z^{-1} \sum_{i=1}^{\infty} \pi_i z^i + \pi_0$$

$$(1 + \rho)G_N(z) = \rho z G_N(z) + z^{-1}(G_N(z) - \pi_0) + \pi_0$$

$$(\rho z^2 - (1 + \rho)z + 1)G_N(z) = (1 - z)\pi_0$$

$$\rho z^2 - (1 + \rho)z + 1 = (1 - z)(1 - \rho z) \Rightarrow$$

$$\begin{aligned}G_N(z) &= \frac{\pi_0}{1 - \rho z}, \\ &= \frac{1 - \rho}{1 - \rho z}\end{aligned}$$

$$\begin{aligned} E[N] &= \left. \frac{dG_N(z)}{dz} \right|_{z=1} \\ &= \left. \frac{1-\rho}{(1-\rho z)^2} \rho \right|_{z=1} \\ &= \frac{\rho}{1-\rho} = \frac{1}{\mu-\lambda} \end{aligned}$$

Laplace Transform

- R.v. X has pdf $f_X(x)$
 - X is non-negative, real value
- The LT of X is:

$$F_X^*(s) \equiv E[e^{-sX}] = \int_0^{\infty} f_X(x)e^{-sx} dx$$

Example

- X : exp. Distr. $f_X(x) = \lambda e^{-\lambda x}$

$$F_X^*(s) = \int_0^{\infty} \lambda e^{-x(\lambda+s)} dx = \frac{\lambda}{\lambda+s}$$

- Moments:

$$E[X] = - \left. \frac{d}{ds} F_X^*(s) \right|_{s=0}$$

$$E[X^i] = (-1)^i \left. \frac{d^{(i)}}{ds} F_X^*(s) \right|_{s=0}$$

Convolution

- X_1, X_2, \dots, X_n are independent rvs with
 $F_{X_1}^*(s), F_{X_2}^*(s), \dots, F_{X_n}^*(s)$

- If $Y = X_1 + X_2 + \dots + X_n$

$$F_Y^*(s) = F_{X_1}^*(s) \cdot F_{X_2}^*(s) \cdot \dots \cdot F_{X_n}^*(s)$$

- If Y is n-th Erlang,

$$F_Y^*(s) = \left(\frac{\lambda}{\lambda + s} \right)^n$$

Z-transform and LT

- X_1, X_2, \dots, X_N are i.i.d. r.v with LT $F_X^*(s)$
- N is r.v. with pgf $G_N(z)$
- $Y = X_1 + X_2 + \dots + X_N$

$$F_Y^*(s) = G_N(F_X^*(s))$$

- If X_i is discrete r.v. with $G_X(z)$, then

$$G_Y(z) = G_N(G_X(z))$$